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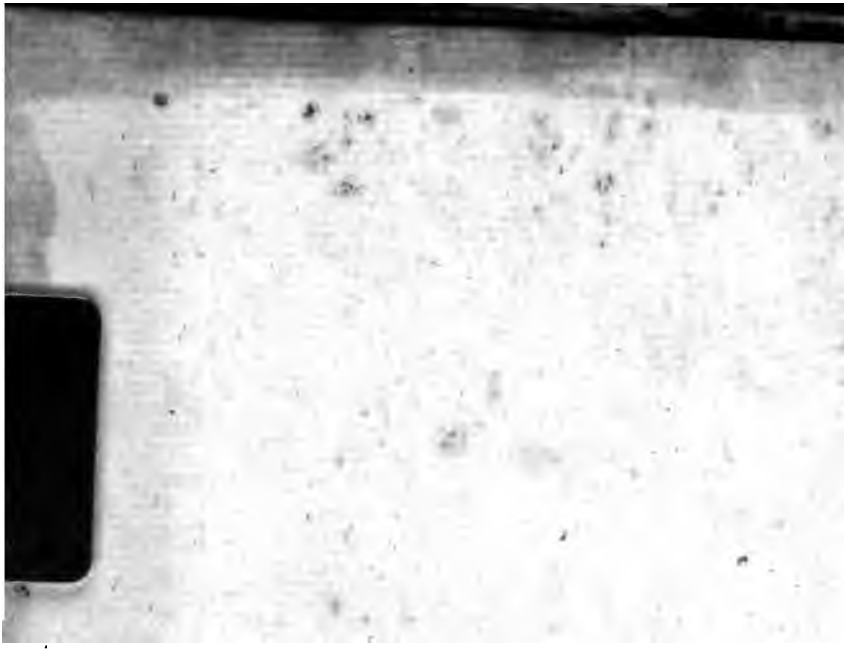

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TRACTS

ON THE

Reſolution of Affected Algebräick Equations

BY

Various Methods of Approximation.

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TRACTS

R.7.16

ON THE

Resolution of Affected Algebraick Equations

BY

DR. HALLEY'S, MR. RAPHSON'S,

AND

SIR ISAAC NEWTON'S,

METHODS OF APPROXIMATION.

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THE PREFACE.

THE principal object of the preſent Collection of Tracts is to explain, and illuſtrate by examples, Mr. Raphſon's and Dr. Halley's Methods of Reſolving Affected Algebräick Equations by Approximation, and to compare theſe two methods with each other, in order to be able to form a judgement of their reſpective merits and determine to which of them we ought to give the preference. With this view I have, in the beginning of the Book, reprinted Dr. Halley's Diſcourſe on this ſubject from the Second Volume of the Collection of Tracts called *Miſcellanea Curioſa*, which was publiſhed in the year 1708. In this Diſcourſe (which is written in a very conciſe and obſcure ſtyle,) Dr. Halley, firſt, gives an accoúnt of what had been done by Mr. *Raphſon* and Monſieur *De Lagny* of Paris with reſpect to this buſineſs of reſolving equations by approximation, and particularly in the reſolution of *pure* equations, or the extraction of the roots of numbers, and then proceeds to deſcribe his own method of reſolving *affected* equations of all degrees

by approximation, which he conſiders as a great improvement of the method that had been given by Mr. Raphſon for the ſame purpoſe. This deſcription is but ſhort, being contained in only two pages, to wit, pages 12 and 13 of the preſent publication; and it is, in my opinion, very obſcure and difficult to underſtand: but it is illuſtrated in the following pages of the Tract, (to wit, pages 14, 15, &c - - - 21,) by being applied to the reſolution of three numeral equations, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, the incompleat biquadratick equation $x^4 - 3x^2 + 75x = 10{,}000$, and the compleat biquadratick equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. This Diſcourſe of Dr. Halley I have endeavoured to explain and illuſtrate in a very long Appendix to it, which extends from page 25 to page 183, and of which the principal contents are as follows:

The firſt 12 pages of this Appendix, to wit, pages 25, 26, &c, - - - - 37, contain a very full deſcription of the grounds of both Dr. Halley's and Mr. Raphſon's methods of reſolving equations by approximation, and, indeed, of all other methods of the ſame kind; which conſiſt of the four following operations; to wit, in the firſt place, to find, (by a few eaſy conjectures and trials, with ſmall numbers conſiſting of one, or two, figures, or by ſome other method ſuggeſted by the conditions of the equation, or of the Problem from which it is derived,) a tolerably near value of x, or the unknown root of the propoſed equation; and, ſecondly, (calling the ſaid near value a,) to put z for the unknown difference between a and the root x; and, if a is leſs than x, and conſequently

conſequently x is equal to $a + z$, to ſubſtitute the binomial quantity $a + z$ inſtead of x in the terms of the original equation; and, if a is greater than x, and conſequently x is equal to $a - z$, to ſubſtitute $a - z$ inſtead of x in the terms of the original equation; by which ſubſtitution the original equation, of which the unknown quantity x was the root, will be transformed into another equation, of which the unknown quantity z (which is much leſs than x, and uſually leſs than a tenth part, and often leſs than a hundredth part, of it,) will be the root; and, thirdly, to omit, or reject from the ſaid ſecond, or transformed, equation, of which z is the root, all the terms that involve any higher powers of z than it's ſquare zz, or than it's ſimple power z; and then, in the 4th and laſt place, to reſolve the remaining equation (which will be either a quadratick equation or a ſimple equation, according as the terms involving zz are retained or rejected,) in the common methods given for that purpoſe. For by theſe operations we ſhall obtain a near value of the ſecond unknown quantity z, and conſequently a near value of $a + z$, or $a - z$, or of the root x of the original equation, which will be nearer to it's true value than the former near value of it, a, was. Theſe operations are the grounds of both Dr. Halley's and Mr. Raphſon's methods of approximation: and the difference between them is only this; to wit, that Dr. Halley retains, in the ſecond, or transformed, equation, of which z is the root, all the terms that involve the ſquare of z as well as thoſe that involve it's ſimple power, and is thereby under the neceſſity of reſolving a quadratick equation in order to obtain the value of z; whereas

Mr. Raphſon rejects all the terms that involve the ſquare of z as well as thoſe that involve z^3, z^4, z^5, and it's other higher powers, and retains only the terms that involve the ſimple power of z, or z itſelf, and by ſo doing is enabled to find the value of z by reſolving only a ſimple equation. The conſequence of this difference is, that the value of z obtained by a ſingle proceſs of Dr. Halley's method will be more exact than the value of it obtained by a ſingle proceſs of Mr. Raphſon's method; but the difficulty of obtaining it is conſiderably greater in the former method than in the latter. How far one of theſe advantages counter-balances the other, and which of the two methods, upon the whole, deſerves to be preferred to the other, can only be determined by a cloſe and carefull compariſon of the two methods with each other in the application of them to the reſolution of the very ſame numeral equations, and by carrying the inveſtigations of the roots of thoſe equations to the ſame, or nearly the ſame, degree of exactneſs, or to the ſame, or nearly the ſame, number of figures, by both the methods. This therefore is what I have endeavoured to do, with reſpect to the three numeral equations above-mentioned, (which have been reſolved by Dr. Halley in the preceeding Tract,) in the ſubſequent pages of this Appendix.

The cubick equation $x^3 - 17x^2 + 54x = 350$ is that which is firſt examined: and the reſolution of it begins in page 37, and ends in page 49, taking up 12 pages. In pages 38 and 39 I ſhew how, by various eaſy reaſonings, conjectures and trials, we may conclude that the true value of x is ſomewhat leſs than the number 15.

And then, putting a for 15, and e (in conformity to Dr. Halley's notation,) for the unknown difference of a and x, I ſubſtitute $a - e$ inſtead of x in the equation $x^3 - 17x^2 + 54x = 350$, and thereby obtain the transformed equation

$$\left\{\begin{array}{l} a^3 - 3a^2e + 3ae^2 - e^3 \\ -17a^2 + 34ae - 17e^2 \\ +54a - 54e \end{array}\right\}$$

$= 350$, which, (by ſubſtituting in it, inſtead of a, a^2, and a^3, their ſeveral numeral values 15, 225, and 3375, and by making the ſubtractions and additions of the terms which are neceſſary to bring all the unknown terms, or terms involving e, to the left-hand ſide of the equation,) becomes $219e - 28e^2 + e^3 = 10$. From this equation I then reject e^3, as being very ſmall in compariſon of $219e$ and $28e^2$, and thereby obtain the quadratick equation $219e - 28e^2 = 10$; which, by a diviſion of all it's terms by 28, is reduced to the equation $\frac{219e}{28} - e^2 = \frac{10}{28}$. And, this equation being reſolved by the proper methods of reſolving quadratick equations, we find that it's two roots are $\frac{219}{56} + \frac{216.427,817\ \&c}{56}$, and $\frac{219}{56} - \frac{216.427,817,\ \&c}{56}$; of which, (by proper reaſonings grounded on our knowledge of the limits between which the original unknown quantity x muſt lie,) we conclude that the leſſer root $\frac{219}{56} - \frac{216.427,817,\ \&c}{56}$ muſt be the value of e that is wanted for our purpoſe. And therefore

we conclude that e, or the difference between 15 and x, will be nearly $= \frac{219}{56} - \frac{216.427,817, \&c}{56}$ $(= \frac{2.572,183, \&c}{56})$ $= 0.045,932$, and conſequently that x, or $a - e$, or $15 - e$, will be, nearly, $= 15 - 0.045,932 = 14.954,068$. Q. E. I.

Having thus gone through the reſolution of the equation $x^3 - 17x^2 + 54x = 350$ by Dr. Halley's method, I proceed to reſolve it by Mr. Raphſon's method, beginning the approximation to the value of x from the ſame number 15, which was taken for a, or it's firſt near value, in the foregoing reſolution of the equation by Dr. Halley's method.

By ſubſtituting $a - e$ inſtead of x in the equation $x^3 - 17x^2 + 54x = 350$ we transform it, as before, into the equation

$$\left\{\begin{array}{r} a^3 - 3a^2e + 3ae^2 - e^3 \\ - 17a^2 + 34ae - 17e^2 \\ + 54a - 54e \end{array}\right\}$$

$= 350$; and, by rejecting from this equation the three terms $3ae^2 - 17e^2 - e^3$, (which involve the ſquare and cube of e,) we reduce it to the ſimple equation $a^3 - 17a^2 + 54a - 3a^2a + 34ae - 54e = 350$, which (by ſubſtituting in it, inſtead of a, a^2, and a^3, their reſpective values 15, 225, and 3375, and by making the proper ſubtractions and additions of the terms, to bring all the unknown terms, or terms involving e, to the left-hand ſide

ſide of the equation,) becomes $219e = 10$. And, this eaſy ſimple equation being reſolved (by dividing both ſides of it by 219, the co-efficient of e,) we have $e = \frac{10}{219} = 0.0456$, and conſequently x, or $a - e$, or $15 - e$, nearly $= 15 - 0.0456 = 14.9544$. Q. E. I.

This proceſs is evidently much ſhorter and ſimpler than the foregoing proceſs by Dr. Halley's method, and gives us the value of x, or the root of the original equation $x^3 - 17x^2 + 54x = 350$, exact to five places of figures, the error of the number 14.9544 beginning only in the laſt figure 4, which in the more accurate value of x is a cypher. This, however, is a very conſiderable degree of exactneſs obtained by a ſingle proceſs of Mr. Raphſon's method, which is much eaſier than the former, or correſponding, proceſs of Dr. Halley's method.

I then proceed, in pages 46, 47, 48, and 49, to find a more exact value of x, or the root of the propoſed equation $x^3 - 17x^2 + 54x = 350$, by means of a ſecond proceſs of Mr. Raphſon's method, taking 14.954, (or the five figures that are exact in the number 14.9544, which was obtained by means of the former proceſs,) for the ground-work of the new proceſs. I therefore ſubſtitute 14.954 inſtead of x in the trinomial quantity $x^3 - 17x^2 + 54x$, and finding that the reſult of this ſubſtitution is $= 349.985,150,664$, which is leſs than the abſolute term 350, I conclude that 14.954 is leſs than the true value of x, and, putting c for 14.954, and f for the unknown difference by which this number falls

ſhort of the true value of x, I ſubſtitute the binomial quantity $c + f$ inſtead of x in the equation $x^3 - 17x^2 + 54x = 350$; by which it is transformed into the equation

$$\left\{\begin{array}{l} c^3 + 3c^2f + 3cf^2 + f^3 \\ -\ 17c^2 - 34cf - 17f^2 \\ +\ 54c + 54f \end{array}\right\} = 350.$$

I then reject from this equation the three terms $3cf^2$, $-17f^2$, and $+f^3$, agreeably to Mr. Raphſon's directions; and the remaining terms form the ſimple equation $c^3 + 3c^2f - 17c^2 - 34cf + 54c + 54f$ nearly $= 350$, which (by ſubſtituting in it's terms, inſtead of c, c^2, and c^3, their ſeveral numeral values, and by making ſuch additions and ſubtractions of the ſeveral terms as are neceſſary to bring all the terms that involve the unknown quantity f to the left-hand ſide of the equation,) produces the ſimple equation $216.430,348 \times f = 0.014,849,336$. And the reſolution of this laſt equation, (by dividing both ſides of it by 216.430,348, or the co-efficient of f,) gives us $f\ (= \frac{0.014,849,336}{216.430,348}) = 0.000,068,610,2$. Therefore x, or $c + f$, or $14.954 + f$, will be nearly $(= 14.954 + 0.000,068,610,2) = 14.954,068,610,2$. Q. E. I.

This number 14.954,068,610,2 is probably true in all it's twelve figures. But it is certainly ſo in the firſt ten figures 14.954,068,61; of which the firſt eight figures 14.954,068, are the ſame as thoſe of the former value of

of x, which was obtained by Dr. Halley's method of approximation. So that one proceſs of Dr. Halley's method of approximation gives us the value of x in the cubick equation $x^3 - 17x^2 + 54x = 350$ exact to eight figures by the reſolution of the quadratick equation $219e - 28e^2 = 10$, which has two roots, of which the leſſer is the value of e wanted for our purpoſe; and two proceſſes of Mr. Raphſon's method of approximation give us the value of the ſame quantity exact to at leaſt ten places of figures by the reſolution of the two ſimple equations $219e = 10$, and $216.430,348 \times f = 0.014,849,336$. The reader muſt now judge for himſelf to which of theſe two methods he will give the preference. For my own part, I give it to Mr. Raphſon's method.

The next equation to be conſidered is the incompleat biquadratick equation $x^4 * - 3x^2 + 75x = 10,000$. This I therefore proceed next to reſolve, firſt, by Dr. Halley's method and afterwards by Mr. Raphſon's, in a very full and diſtinct manner, in pages 49, 50, 51, &c--65.

And, in the firſt place, in pages 50 and 51 I find, by means of ſome eaſy conjectures and trials, that the value of x in this equation will be greater than 9, but leſs than 10, and nearer to 10 than to 9. I therefore make $a = 10$, and, putting e for the difference between a and x, ſubſtitute $a - e$ inſtead of x in the propoſed equation $x^4 * - 3x^2 + 75x = 10,000$; which is thereby transformed into the equation

$$\left\{\begin{array}{r} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\}$$

$= 10{,}000$. From this equation I then reject, according to Dr. Halley's directions, the two terms $- 4ae^3 + e^4$, which involve in them the cube and fourth power of e, and thereby reduce the ſaid equation to the quadratick equation

$$\left\{\begin{array}{r} a^4 - 4a^3e + 6a^2e^2 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e \end{array}\right\} = 10{,}000,$$

which (by ſubſtituting in it, inſtead of a, a^2, a^3, and a^4, their ſeveral values 10, 100, 1000, and 10,000, and making the ſeveral additions and ſubtractions of the terms which are neceſſary to bring all the terms that involve the unknown quantity e to the left-hand ſide of the equation,) becomes changed into the numeral equation $4015e - 597e^2 = 450$, and (by dividing all the terms by 597, or the co-efficient of e^2,) into the equation $6.725{,}293 \times e - e^2 = 0.753{,}769$, which is a quadratick equation duely prepared for reſolution.

This quadratick equation I then proceed to reſolve, and find that it has two roots, to wit, $3.362{,}646 + 3.248{,}633$ and $3.362{,}646 - 3.248{,}633$, of which (by conſidering the limits of the magnitude of x, or $a - e$, which is known to be leſs than 10, but greater than 9, which makes it neceſſary for e to be leſs than 1,) I find that

that the leſſer root 3.362,646 — 3.248,633 muſt be that which will be ſuited to our purpoſe, or will be nearly equal to the difference between a, or 10, and x. I therefore conclude that e will be nearly = 3.362,646 — 3.248,633 = 0.114,013, and therefore that x, or $a - e$, or $10 - e$, will be nearly = 10 — 0.114,013 or (neglecting the three laſt figures 013, as probably not exact) = 10 — 0.114 = 9.886. Q. E. I.

This number 9.886 appears, upon a trial of it, to be very nearly equal to, but ſomewhat leſs than, the true value of x in the propoſed equation $x^{4*} - 3x^2 + 75x = 10{,}000$. In order therefore to find a more accurate value of x, I put e for 9.886, and f for it's unknown difference from the true value of x, and ſubſtitute the binomial quantity $e + f$ inſtead of x in the equation $x^{4*} - 3x^2 + 75x = 10{,}000$; by means of which ſubſtitution the ſaid equation is transformed into the equation

$$\left\{\begin{array}{r} e^4 + 4e^3f + 6e^2f^2 + 4ef^3 + f^4 \\ - 3ee - 6ef - 3f^2 \\ + 75e + 75f \end{array}\right\}$$

= 10,000, of which the ſmall quantity f is the root. I then reject from this equation the two terms $4ef^3$ and f^4, which reduces it to the quadratick equation

$$\left\{\begin{array}{r} e^4 + 4e^3f + 6e^2f^2 \\ - 3ee - 6ef - 3f^2 \\ + 75e + 75f \end{array}\right\} = 10{,}000;$$

and (by ſubſtituting in this laſt equation, inſtead of e, e^2, e^3, and e^4, their ſeveral values 9.886 and it's ſquare, cube,

cube, and fourth power, and by making the ſeveral additions and ſubtractions of the terms which are neceſſary to bring all the terms that involve the unknown difference f to the left-hand ſide of the equation,) this equation is converted into the numeral equation 3880.437,593,824 $\times f$ + 583.397,976 $\times ff$ = 0.010,480,863,984; and this equation (by dividing all it's terms by 583.397,976, or the numeral co-efficient of ff) is further reduced to the equation 6.651,441,646 $\times f$ + ff = 0.000,017,965,204,569,033, which is a quadratick equation duely prepared for reſolution. This equation (which has only one root,) is then reſolved, but not without a great deal of laborious calculation; and it's root f is found to be = 0.000,002,700. Therefore x, or $e + f$, or 9.886 + f, is nearly (= 9.886 + 0.000,002,700) = 9.886,002,700; that is, the root of the propoſed biquadratick equation is nearly equal to 9.886,002,700.

Q. E. I.

Dr. Halley makes this root to be nearly equal to 9.886,260,393,649,5. But I believe he muſt have made ſome miſtake in his calculation, becauſe the other value of x here found, to wit, 9.886,002,700, is confirmed by the ſecond reſolution which I have made of this equation by Mr. Raphſon's method. This ſecond reſolution is as follows:

Let 10 be taken for a, or the firſt near value of x in the equation $x^{4} * * - 3x^{2} + 75x = 10,000$, as it was in the foregoing reſolution of that equation by Dr. Halley's method; and let $a - e$ be ſubſtituted inſtead of x in

in the three terms x^4, $3x^2$, and $75x$. Then will the transformed equation reſulting from ſuch ſubſtitution be (as before)

$$\left\{\begin{array}{l} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ -3a^2 + 6ae - 3e^2 \\ +75a - 75e \end{array}\right\}$$

$= 10{,}000$; which, by rejecting the four terms $6a^2e^2 - 3e^2 - 4ae^3 + e^4$ according to the directions of Mr. Raphſon, will be reduced to the ſimple equation

$$\left\{\begin{array}{l} a^4 - 4a^3e \\ -3a^2 + 6ae \\ +75a - 75e \end{array}\right\} = 10{,}000.$$

And this laſt equation will (by ſubſtituting in it's terms, inſtead of a, a^2, a^3, and a^4, their ſeveral values 10, 100, 1000, and 10,000, and by making ſuch additions and ſubtractions of the terms as are neceſſary in order to bring all the terms that involve the unknown quantity e to the left-hand ſide of the equation,) be converted into the ſhort numeral equation $4015e = 450$. And the reſolution of this laſt equation (by dividing both ſides of it by 4015, the co-efficient of the unknown quantity e,) gives us e $(= \frac{450}{4015}) = 0.112$. Therefore x, or $a - e$, or $10 - e$, will be nearly $= 10 - 0.112 = 9.888$; or the ſecond near value of x in the propoſed equation $x^4 * - 3x^2 + 75x = 10{,}000$, obtained by this firſt proceſs of Mr. Raphſon's method of approximation, will be 9 888. Q. E. I.

The

poſed equation $x^{4}* - 3x^{2} + 75x = 10{,}000$, but with an omiſſion of the terms that involve either g^{2}, g^{3}, or g^{4}, agreeably to Mr. Raphſon's directions; and by this ſubſtitution I transform the ſaid equation into a ſimple equation, which, when the terms of it are properly ranged, becomes the equation $3880.449{,}261{,}795{,}383{,}204 \times g = 0.028{,}313{,}570{,}294{,}077{,}144{,}01$. And, this laſt equation being reſolved by dividing both ſides of it by the co-efficient of g, we ſhall have $g = \frac{0.028{,}313{,}570{,}294{,}077{,}144{,}01}{3880.449{,}261{,}795{,}383{,}204} = 0.000{,}007{,}296{,}467$, and conſequently x, or $9.886{,}01 - g$, $= 9.886{,}01 - 0.000{,}007{,}296{,}467 = 9.886{,}002{,}703{,}533$; that is, the fourth near value of x in the propoſed equation $x^{4}* - 3x^{2} + 75x = 10{,}000$, obtained by this third proceſs of Mr. Raphſon's method of approximation, will be $9.886{,}002{,}703{,}533$. Q. E. I.

Of this number $9.886{,}002{,}703{,}533$ it is almoſt certain that (if no miſtakes have been made in the calculation,) the firſt ten figures $9.886{,}002{,}703$ are exact. And, as the firſt nine of them, to wit, $9.886{,}002{,}70$, are the ſame with the firſt nine figures of the laſt value of x that had been found by the ſecond proceſs of Dr. Halley's method of approximation, to wit, $9.886{,}002{,}700$, I think we may be confident that, at leaſt, theſe nine figures muſt be exact, and therefore that Dr. Halley's number $9.886{,}260{,}393{,}649{,}5$, given above in page 15 for a moſt accurate value of x ſcarce exceeding the truth by 2 in the laſt figure, muſt be erroneous.

It

It ſeems to me, from the compariſon of the two methods of approximation in this example, as well as in the laſt, that Mr. Raphſon's method (which proceeds by the reſolution of only ſimple equations) is ſo much ſimpler and eaſier, both to underſtand and to practiſe, than Dr. Halley's method (which proceeds by the reſolution of quadratick equations,) that, notwithſtanding the advantage of the latter method in giving us more new figures of the root ſought exactly, in every ſingle proceſs of it, than are given by a ſingle proceſs of the former method, it very much deſerves to be preferred to the latter, or Dr. Halley's, method.

I next proceed to Dr. Halley's third example, which is the biquadratick equation $14{,}937x - 1998x^2 + 8cx^3 - x^4 = 5000$. This equation he juſtly conſiders as a very difficult one, becauſe it is of that form which admits of four different affirmative roots, and becauſe the co-efficients of x and x^2, to wit, 14,937 and 1998, are very great numbers in compariſon of the reſolvend, or abſolute term, 5000. And in truth this equation has four affirmative roots, to wit, the decimal fraction 0.350,986,045,866,06 &c, the mixt number 12.756,441, 794,480,744,022, &c, the mixt number 32.060,290, &c, and the mixt number 34.832,280, &c. But of theſe roots Dr. Halley finds only the ſecond, or leaſt but one, to wit, 12.756,441,794,480,744,022, &c. This root therefore I, in page 65, proceed to inveſtigate, firſt, by Dr. Halley's method of approximation, and afterwards by Mr. Raphſon's.

In the firſt place I conjecture that x is nearly equal to 10, and I ſubſtitute 10 inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and I find the value of the ſaid quantity reſulting from that ſubſtitution to be 19,570, which is almoſt four times as great as 5000, or the abſolute term of the propoſed equation. I then form a ſecond conjecture about it's magnitude, and ſuppoſe it to be, nearly, = 12, and ſubſtitute 12 inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and find the reſult of this ſubſtitution to be 9,036. This reſult, though much leſs than the former reſult 19,570, is yet much greater than the abſolute term 5000.

By theſe two conjectures and trials I find that, while x increaſes from 10 to 12, the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ decreaſes from 19,570 to 9,036. I therefore ſuppoſe that, if x were to increaſe further from 12 to 13, the ſaid quadrinomial quantity would decreaſe further from 9,036 to ſome leſſer number. And ſo, upon trial, I find it to do. For, if x is = 13, the ſaid quadrinomial quantity will be = 3,718, which is leſs than the abſolute term 5000. I therefore now conclude with certainty that x muſt be greater than 12, but leſs than 13.

Further, in order to obtain a ſtill nearer value of x, I make uſe of the following conjectural, but very probable, and (as appears upon trial,) very uſeful, ſuppoſition. Since, when x is = 12, the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is = 9,036, and,

when

when x is equal to the root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, the ſaid quadrinomial quantity is $= 5000$, and, when x is $= 13$, the ſaid quadrinomial quantity is $= 3,718$, it ſeems probable that the difference of the firſt and third of the firſt three quantities 12, x, and 13, to wit, the difference $13 - 12$, or 1, will be to the difference of the firſt and ſecond of theſe quantities, to wit, $x - 12$, in, nearly, the ſame proportion as the difference of the firſt and third of the latter three quantities 9,036, 5000, and 3,718, which correſpond to the firſt and third quantities, 12 and 13, of the firſt ſet, is to the difference of the firſt and ſecond of the latter three quantities 9,036, 5000, and 3,718, which correſpond to the firſt and ſecond quantities, 12 and x, of the firſt ſet; that is, that 1 will be to $x - 12$ in, nearly, the ſame proportion as $9,036 - 3,718$, or 5,318, is to $9,036 - 5000$, or 4,036, and conſequently that $x - 12$ will be nearly $= \frac{1 \times 4,036}{5,318} = \frac{4,036}{5,318} = 0.7$ &c, and therefore that x will be nearly $= 12.7$.

Having thus obtained 12.7 for a firſt near value of x that is ſufficiently near to it's true value to become the baſis, or ground-work, of a farther approximation to it's true value by either Dr. Halley's, or Mr. Raphſon's method of approximation, I proceed to make uſe of it for obtaining a more exact value of x by Dr. Halley's method of approximation. And for that purpoſe I begin by ſubſtituting it inſtead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover

whether the value of the ſaid quadrinomial quantity reſulting from ſuch ſubſtitution will be greater or leſs than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now the reſult of this ſubſtitution is 5298.6559. And therefore it appears that, while x increaſes from 12.7 to 13, the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will decreaſe from 5298.6559 to 3,718. And therefore, when x is of ſome intermediate value between 12.7 and 13, the ſaid quadrinomial quantity will be equal to the abſolute term 5000, or, in other words, the true value of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be greater than 12.7, but leſs than 13. We muſt therefore now make $12.7 = a$, and $12.7 + e$, or $a + e$, $= x$, and ſubſtitute $a + e$ inſtead of x in the propoſed equation, but with an omiſſion of the terms that involve either e^3 or e^4. This ſubſtitution will produce the tranſformed equation

$$\left\{\begin{array}{llll} 14{,}937a & + 14{,}937e & & \\ - 1998a^2 & - 3996ae & - 1998e^2 & \\ + 80a^3 & + 240a^2e & + 240ae^2 & \\ - a^4 & - 4a^3e & - 6a^2e^2 & \end{array}\right\}$$

$=$, nearly, 5000. And (by ſubſtituting in this equation, inſtead of a, a^2, a^3, and a^4, their ſeveral values, to wit, 12.7, and it's ſquare, cube, and fourth power, and making the ſeveral additions and ſubtractions of the terms which are neceſſary to bring all the terms that involve

involve the unknown quantity e to the left-hand ſide of the equation, with only a known number on it's right-hand ſide,) we further reduce it to the equation $5296.132 \times e - 82.26 \times e^2 = 298.6559$, and (by dividing all the terms by 82.26, or the co-efficient of e^2,) to the equation $64.382,834,9 \times e - e^2 = 3.630,633,357,646,48$, which is a quadratick equation properly prepared for reſolution.

This quadratick equation (which has two roots,) is then reſolved, but not without a great deal of laborious calculation: and it's two roots are found to be 32.191,417,4 + 32.134,976,5, and 32.191,417,4 − 32.134,976,5, of which it is evident that the former root (which is greater than 64) cannot be the value of e that is ſuited to the preſent purpoſe, or is nearly equal to the difference between 12.7 and the true value of x, which we know to be leſs than 13; and therefore we conclude that the ſecond root, 32.191,417,4 − 32.134,976,5, is the root that is to be adopted on this occaſion. We therefore conclude that e, or the difference between x and 12.7 will be nearly = 32.191,417,4 − 32.134,976,5, or 0.056,440,9, and conſequently that x, or $a + e$, or $12.7 + e$, will be, nearly, (= 12.7 + 0.056,440,9) = 12.756,440,9. Q. E. I.

This value of x is exact in the firſt ſeven figures 12.756,44, it's more accurate value, as computed both by Dr. Wallis and Dr. Halley, being 12.756,441,794,480,744,02.

Dr. Halley on this occaſion points out a correction to be made to the value of x juſt now obtained, to wit, the number 12.756,440,9, without entering upon a compleat ſecond proceſs of his method of approximation, and tells us that we may, by this correction, find the value of x to be $=$ 12.756,441,794,48, or to thirteen places of figures, all exact. This correction I have endeavoured to explain and to put in practice, in pages 73, 74, and 75. But I am not ſure that I have perfectly underſtood it; and the value of x reſulting from my application of it does not agree with the value aſſigned by Dr. Halley, to wit, 12.756,441,794,48, but differs from it in ſome of the latter figures, being 12.756,441,794,387. The reaſon of this difference I do not know. But, as I do not approve of this correction for the reaſons alledged in art. 34, pages 76 and 77, I ſhall ſay no more of it in this place, but ſhall proceed to give an account of the compleat ſecond proceſs of Dr. Halley's method of approximation which I have gone through in pages 77, 78, 79, - - - 82, and by which I find the more accurate value of x to be 12.756,441,794,480,744,022,60; of which I believe the firſt 20 figures, to wit, 12.756,441, 794,480,744,022, to be exact.

In order to begin the ſaid ſecond proceſs, I ſubſtitute 12.756,44 inſtead of x in the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$; and I find the value of the ſaid quantity reſulting from ſuch ſubſtitution to be 5000,009,486,644,489,476,503,04; which is a very little greater than 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 =$ 5000.

5000. And I thence conclude, for the reaſons given in art. 29, that the number 12.756,44 is ſomewhat leſs than the true value of x.

I therefore put c for the number 12.756,44, and f for it's unknown difference from the true value of x, and ſubſtitute the binomial quantity $c + f$ inſtead of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$; by which ſubſtitution the ſaid equation is transformed into the equation

$$\left\{\begin{array}{l} 14{,}937c + 14{,}937f \\ -\ 1998c^2 - 3996cf - 1998ff \\ +\ 80c^3 + 240c^2f + 240cff + 80f^3 \\ -\ c^4 - 4c^3f - 6c^2ff - 4cf^3 - f^4 \end{array}\right\}$$

$= 10{,}000$, or (omitting the three terms $80f^3$, $4cf^3$ and f^4) into the equation

$$\left\{\begin{array}{l} 14{,}937c + 14{,}937f \\ -\ 1998c^2 - 3996cf - 1998ff \\ +\ 80c^3 + 240c^2f + 240cff \\ -\ c^4 - 4c^3f - 6c^2ff \end{array}\right\}$$

$=$, nearly, 10,000. And, if we ſubſtitute in this laſt equation, inſtead of c, c^2, c^3, and c^4, their ſeveral values, to wit, 12.756,44, and it's ſquare, cube, and fourth power, and make the ſeveral additions and ſubtractions that are neceſſary to bring all the terms that involve the unknown quantity f to the left-hand ſide of the equation, this laſt equation will be converted into the equation

tion 5286.568,162,865,159,936 × f — 87.184,431,158,4 × ff = 0.009,486,644,589,476,503,04, and finally (by dividing all the terms of the laſt equation by 87.184,431,158,4, the co-efficient of ff,) into the equation 60.636,607,851,008,642,270,08 × f — ff = 0.000,108,811,221,951,323,000,352,097,689,829,824,386,0 &c, which is a quadratick equation duely prepared for reſolution.

This equation is then reſolved in page 82, but not without a great deal of laborious calculation; and it's leſſer root (which is that which is equal to the difference between 12.756,44 and x) is found to be 0.000,001,794,480,744,022,60. Therefore x, or $c + f$, or 12.756,44 + f, will be, nearly, (= 12.756,44 + 0.000,001,794,480,744,022,60) = 12.756,441,794,480,744,022,60; that is, the value of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be very nearly equal to 12.756,441,794,480,744,022,60. Q. E. I.

I then proceed in pages 83, 84, 85, &c - - - - 96, to find the value of the ſame root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ by Mr. Raphſon's method of approximation.

In making this approximation I begin with the ſame firſt near value of x with which I began the former approximation by Dr. Halley's method, to wit, the number 12.7, putting a = 12.7, and $x = a + e$ = 12.7 + e, and ſubſtituting the binomial quantity 12.7 + e inſtead

of

of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, but with an omiſſion of all the terms that involve either e^2, e^3, or e^4. This ſubſtitution produces the ſimple equation $14{,}937a - 1998a^2 + 80a^3 - a^4 + 14{,}937e - 3996ae + 240a^2e - 4a^3e = $, nearly, 5000. And this laſt equation, being further reduced (by ſubſtituting in it's terms inſtead of a, a^2, a^3, and a^4 their ſeveral numeral values, to wit, 12.7, and it's ſquare, cube, and fourth power, and by making the ſeveral additions and ſubtractions which are neceſſary to bring all the terms involving e to the left-hand ſide of the equation, and to conſolidate them into one term) becomes $5296.132 \times e = 298.6559$; the reſolution of which gives $e \left(= \frac{298.6559}{5296.132}\right) = 0.056{,}39$. Therefore x, or $a + e$, or $12.7 + e$, will be $(= 12.7 + 0.056{,}39) = 12.756{,}39$; that is, the ſecond near value of x, obtained by this firſt proceſs of Mr. Raphſon's approximation, will be 12.756,39. Q. E. I.

Of this number 12.756,39 the firſt five figures 12.756 are exact, the more accurate value of x being 12.756,441,794,480, &c.

I then ſubſtitute 12.756,39 inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and find the reſult of the ſaid ſubſtitution to be $= 5000.273{,}895{,}270{,}591{,}690{,}609{,}59$; which is ſomewhat greater than 5000, or the abſolute term of the propoſed equation. Therefore, for the reaſons before given, I conclude that 12.756,39 muſt be leſs than the true value of x. I therefore put $c = 12.756{,}39$, and f for the unknown quantity by which 12.756,39 falls ſhort of the true

true value of x, and ſubſtitute $c + f$ inſtead of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, but with an omiſſion of all the terms that involve either f^2, f^3, f^4. And the transformed equation thence ariſing is

$$\left\{\begin{array}{rrr} & 14{,}937c & + 14{,}937f \\ - & 1998c^2 & - 3996cf \\ + & 80c^3 & + 240c^2f \\ - & c^4 & - 4c^3f \end{array}\right\}$$

nearly, $= 5000$; in which if we ſubſtitute for c, c^2, c^3, and c^4, their ſeveral numeral values, to wit, 12.756,39, and it's ſquare, cube, and fourth power, and afterwards make the ſeveral additions and ſubtractions of the terms which are neceſſary to conſolidate the four terms involving the unknown quantity f into one term, and to conſolidate all the known quantities in the equation into one term, and bring it to the oppoſite ſide of the equation, we ſhall obtain the ſimple equation $5286.576{,}881{,}150{,}968{,}476 \times f = 0.273{,}805{,}270{,}591{,}690{,}609{,}59$, by the reſolution of which we ſhall have $f \left(= \frac{0.273{,}805{,}270{,}591{,}690{,}609{,}59}{5286.576{,}881{,}150{,}968{,}476}\right) = 0.000{,}051{,}792$. Therefore $c + f$, or $12.756{,}39 + f$, will be $(= 12.756{,}39 + 0.000{,}051{,}792) = 12.756{,}441{,}792$; that is, the third near value of x, which is obtained by this ſecond proceſs of Mr. Raphſon's method of approximation, will be 12.756,441,792. Q. E. I.

Of this number 12.756,441,792 the firſt ten figures, 12.756,441,79, are exact, the more accurate value of x

being 12.756,441,794,480,744,02, as has been obſerved before. But this greater degree of exactneſs may be attained by carrying this approximation by Mr. Raphſon's method one ſtep further; which may be done as follows:

If the number 12.756,441,792, obtained by the laſt proceſs, be ſubſtituted inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, the value of the ſaid quantity reſulting from ſuch ſubſtitution will be 5000.000,013,114,621,596,455,067,255,345,516,642,304; which is a little greater than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore, for the reaſons before given, 12.756,441,792 muſt be ſomewhat leſs than the true value of x. I therefore put $d = 12.756{,}441{,}792$, and ſuppoſe x to be equal to $d + g$, or to 12.756,441,792 + g, and ſubſtitute that binomial quantity, inſtead of x, in the propoſed equation, but with an omiſſion of all the terms that involve either g^2, g^3, or g^4. And the tranſformed equation thence obtained, after it's terms are properly arranged, becomes the ſimple equation $5{,}286.567{,}850{,}393{,}729{,}132{,}515{,}900{,}260{,}352 \times g = 0.000{,}013{,}114{,}621{,}596{,}455{,}067{,}255{,}345{,}516{,}642{,}304$, as is ſhewn in pages 93, 94, and 95. Therefore g is $(= \frac{0.000{,}013{,}114{,}621{,}596{,}455{,}067{,}255{,}345}{5286.567{,}850{,}393{,}729{,}132{,}515{,}900{,}262{,}352}) = 0.000{,}000{,}002{,}480{,}744{,}022{,}88$; and conſequently x, or 12.756,441,792 + g, will be = 12.756,441,792 + 0.000,000,002,480,744,022,88 = 12.756,441,794,480,744,022,88; that is, the fourth near value of x, which is obtained by this

this third proceſs of Mr. Raphſon's method of approximation, will be 12.756,441,794,480,744,022,88.

Q. E. D.

This value of x (which has been obtained by means of theſe three proceſſes of Mr. Raphſon's method of approximation,) agrees with the value of it above-obtained by means of two proceſſes of Dr. Halley's method of approximation, to wit, 12.756,441,794,480,744,022,60, in the firſt twenty figures 12.756,441,794,480,744,022; and therefore theſe twenty figures are probably exact, and the true value of x is greater than 12.756,441,794,480,744,022, but leſs than 12.756,441,794,480,744,023.

Both theſe methods of attaining the value of x in this equation to this great degree of exactneſs have been attended with a great deal of labour; but I have found the labour neceſſary to the three proceſſes of Mr. Raphſon's method conſiderably leſs than the labour neceſſary to the two proceſſes of Dr. Halley's method. The readers, who ſhall have gone carefully through all the operations of both the methods, will be able to judge for themſelves which of the two methods is the clearer and eaſier, and deſerves, upon the whole, to be preferred to the other.

The reſolutions of the three foregoing equations $x^3 - 17x^2 + 54x = 350$, $x^{4*} - 3x^2 + 75x = 10{,}000$, and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, (which Dr. Halley had produced as examples of his method of approximation,) having been compleated in page 96, I proceed in page 97 to mention, and explain at large, another

another method of reſolving high affected equations by approximation, which is totally different from both the former methods. This method, I think, may with propriety be called *The Differential Method of Approximation*; becauſe it is grounded on a very uſeful property of the differences of the root x of any propoſed equation and any two near values of x. This property may be thus deſcribed. Let the capital letter A be put for the abſolute term of the equation, which is the value of the compound quantity which forms the left-hand ſide of the equation, when the true value of x is ſubſtituted inſtead of x in the ſaid compound quantity. Let b be a near value of x, (the nearer the better,) either greater or leſs than it's near value; and let c be another near value of x, either greater or leſs than it's true value. And let the capital letter B denote the value of the compound quantity which forms the left-hand ſide of the equation, when b is ſubſtituted inſtead of x in the terms of the ſaid compound quantity; and let the capital letter C denote the value of the ſaid compound quantity when c is ſubſtituted in it's terms inſtead of x; ſo that the three quantities B, A, and C ſhall be the values of the ſaid compound quantity correſponding to the three quantities b, x, and c, reſpectively, or reſulting from the ſubſtitution of b, x, and c in the ſaid compound quantity, reſpectively. Theſe things being ſuppoſed, the property of the differences of x and it's two near values b and c is as follows. "The difference of b and "c, the firſt and third of the three former quantities "b, x, and c, will be to the difference of b and x, the "firſt and ſecond of the ſaid three quantities, in, nearly,

"the

" the ſame proportion as the difference of B and C, the " firſt and third of the three latter quantities B, A, and " C, is to the difference of B and A, the firſt and ſecond " of the ſaid three latter quantities."

By means of this proportion the value of x may be derived to a conſiderable degree of exactneſs from the quantities b, c, B, A, and C, which are all known quantities.

It was by means of this proportion that, in ſeeking a, or the firſt near value of x in the laſt-mentioned biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, in order to make it the baſis of a proceſs of Dr. Halley's method of approximation, when we had found that 12 was ſomething leſs, and that 13 was ſomething greater, than the true value of x in that equation, and that, if 12 was ſubſtituted inſtead of x in the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$, the reſult would be $= 9,036$, and that, if 13 was ſubſtituted inſtead of x in the ſame compound quantity, the reſult would be $= 3,718$, we diſcovered that 12.7 would be much nearer to the true value of x than either 12 or 13. For in that caſe the three firſt quantities b, x, and c were 12, x, and 13, and the three latter quantities B, A, and C, correſponding to them, were 9,036, 5000, and 3,718, and we ſuppoſed $13 - 12$ to be to $x - 12$ in nearly the ſame proportion as $9,036 - 3,718$ is to $9,036 - 5000$, or 1 to be to $x - 12$ in nearly the ſame proportion as 5,318 is to 4,036; whence it followed that

that $x - 12$ would be nearly $= \frac{4,036}{5,318} = 0.7$ &c, and conſequently that x would be nearly equal to 0.7 &c + 12, or 12.7.

And, if we had carried the diviſion of 4,036 by 5,318 to two figures more in the quotient, we ſhould have found $x - 12$ to be nearly $= 0.758$, and conſequently x to be nearly $= 12.758$, of which the four firſt figures 12.75 are exact, and the fifth figure 8 is but a little too great, the more accurate value being 12.756,441, &c. So that we might by this means have obtained the four figures 12.75, which are all exact, for the value of a, or of the firſt near value of x, which was to be made the baſis, or ground-work, of a further approach to the true value of x by a proceſs of Dr. Halley's method of approximation. This property therefore of the differences of x and it's two near values b and c appears from this example to be highly uſeful in obtaining a much nearer value of the root x than either of it's two former near values b and c, or, in this inſtance, 12 and 13, which we had obtained before.

This property of the differences of x and it's two near values b and c takes place with reſpect to three logarithms that are nearly equal to each other. For it is the ground of the method by which, when three numbers, which are nearly equal to each other, are given, and the logarithms of the greateſt and leaſt of them are known, the logarithm of the middle number is derived from the known logarithms of the two extreme numbers, and from the three numbers themſelves. For in that caſe, if the

the three numbers be called B, A, and C, and the logarithms of B and C are known, but that of A is unknown, it is cuſtomary to make the following proportion, in order to diſcover the logarithm of A; to wit, log. B — log. C is to log. B — log. A :: B — C : B — A; whence it follows that log. B — log. A will be = $\overline{\text{log. B} - \text{log. C}} \times \frac{\overline{B - A}}{B - C}$, and (adding log. A to both ſides) log. B will be = $\overline{\text{log. B} - \text{log. C}} \times \frac{B - A}{B - C}$ + log. A, and conſequently log. A will be = log. B $- \sqrt{\frac{B - A}{B - C}} \times \overline{\text{log. B} - \text{log. C}}$. Indeed the known application of this proportion to the finding of the logarithms of ſuch intermediate numbers between other numbers whoſe logarithms are known, was the circumſtance that firſt ſuggeſted to me the idea of applying it to the finding of a nearer value of the root x of any propoſed equation when we are already poſſeſſed of two contiguous near values of it, as b and c, and of the reſults of the ſubſtitution of the ſaid near values in the compound quantity that forms the left-hand ſide of the equation.

This *Differential Method* of approximating to the roots of affected equations is explained pretty fully, and illuſtrated by an example, in the Scholium in pages 97, 98, 99, &c - - - 109. And it is there ſhewn that it gives us about as many new figures of the value of x exact as are obtained by one proceſs of Mr. Raphſon's method of

of approximation. However, I think, Mr. Raphſon's method is, upon the whole, ſomewhat preferable to it, for the reaſons aſſigned in art. 50, pages 105, 106, and 107; and I would therefore recommend this differential method to be uſed only in the firſt part of the inveſtigation of x, or the root of any propoſed equation, while we are endeavouring to find a, or the firſt near value of x, (which is to be made the baſis of a nearer approach to the true value of it by Mr. Raphſon's method of approximation,) to a conſiderable degree of exactneſs, as, for example, to three or four places of figures. For in this way, I believe, this differential method will be found extremely uſeful.

After the Scholium concerning the differential method of approximation (which ends in page 109,) I have inſerted ſome obſervations of Mr. Raphſon on Monſieur de Lagny's and Dr. Halley's claims to the merit of having invented their methods of reſolving equations by approximation, which ſhew them to be only branches, or modifications, of his method, which he had conſidered with attention and deliberately rejected, as being leſs ſimple and convenient in practice than his own method. Theſe obſervations are contained in pages 109 and 110.

In page 111 I enter upon an inquiry, whether it may not ſometimes, in extracting the ſquare-root of a number to many places of figures, (as was neceſſary in reſolving the quadratick equations that occurred in the reſolution of the three foregoing equations by Dr. Halley's method of approximation,) be convenient to proceed by ſome

method of approximation rather than by the common method of extracting the ſquare-root. This inquiry extends to page 132; and the reſult is, "that, from all "the trials I have there made of different expreſſions for "approximating to the value of the ſquare-root of a given "number N, or $aa + b$, I am inclined to conclude that, "in performing the extractions of the ſquare-roots of "given numbers which are neceſſary in Dr. Halley's "method of reſolving high affected equations, it will "almoſt always be found eaſier and more convenient to "proceed by the common method of extracting them "than to have recourſe to either of the three expreſſions "found for that purpoſe in the courſe of this inquiry, "not excepting even Mr. Raphſon's ſimple expreſſion "$a + \frac{b}{2a}$, which is very much to be preferred to the "two others."

I then in pages 133, 134, &c reſume the conſideration of the three foregoing equations which Dr. Halley has adduced in his Tract as examples of his method of reſolving equations by approximation, and I inquire whether theſe equations may not have ſome other roots beſides thoſe that have been inveſtigated by Dr. Halley and in the Appendix to his Diſcourſe. And I ſhew, firſt, that the firſt of theſe equations, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, cannot have any other root beſides that above inveſtigated, to wit, 14.954,068,61. This is ſhewn by two different methods, in pages 133, 134, 135, &c, - - 142. And I then ſhew that the biquadratick equation $x^4 - 3x^3 + 75x = 10,000$ cannot

cannot have any other root beſides that above inveſtigated, to wit, 9.886,002,70. This is ſhewn in pages 142, 143, and 144. And I then proceed to examine the third and laſt of the ſaid equations, to wit, the bi-quadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and to inquire, firſt, whether this equation can have any other root greater than the root 12.756,441,794, &c which we have already found, and ſecondly, whether it can have any other root leſs than the ſaid root already found. The former of theſe inquiries begins in page 144, and ends in page 152; and it contains a clear and regular proof that this equation will have two roots that are greater than the root 12.756,441,794, &c, of which the leſſer will be ſomething leſs, and the greater will be ſomething greater, than the number 33.42. The ſecond inquiry begins in page 152, and ends in page 155; and it contains a proof that this equation will alſo have a fourth root, that will be leſs than the root 12.756,441,794, &c already found. And, in pages 156, 157, 158, it is ſhewn that this fourth root will be nearly = to the decimal fraction 0.35; and, in pages 158, 159 and 160, it is further ſhewn by means of only one proceſs of Mr. Raphſon's method of approximation, that the firſt ſix figures of the value of this fourth root will be 0.350,987. A ſecond proceſs of that method of approximation would have given us the decimal fraction 0,350,987,045,856,06, for it's more accurate value. I then in pages 161, 162, &c, . . . 175, inveſtigate to a moderate degree of exactneſs the two greateſt roots of this equation, which had been before found to be not very different from the number 33.42; and I find that the leſſer of thoſe roots

will be = 32.060,290, &c, and that the greater of them will be = 34.832,280, &c. And thus the reſolutions of all the three equations $x^3 - 17x + 54 = 350$, $x^4 - 3x^2 + 75x = 10{,}000$, and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, which Dr. Halley had produced as examples of his method of reſolving affected equations by approximation, are rendered quite compleat and ſatisfactory.

The remaining eight pages 176, 177, 178, &c, - - 183, of the Appendix to Dr. Halley's Tract above-mentioned, contain no Algebräick calculations, but are employed, firſt, in deſcribing the circumſtance that forms the difference between the accurate methods of reſolving Algebräick equations and the methods of reſolving them by approximation, and, ſecondly, in giving an account of the principal methods of reſolving Algebräick equations by approximation that have been publiſhed, and of the times of their publication. The former of theſe points is treated of in pages 176, 177, and 178, and the latter in pages 178, 179, 180, &c - - 183.

The next Tract in this Collection, after this long Appendix to Dr. Halley's Diſcourſe, contains a very full exhibition of Dr. Wallis's Solution of a very difficult Arithmetical Problem that had been propoſed to him by Colonel *Silas Titus*, a perſon of great note in the reign of King Charles the Second, to whom he had at one time had the honour of being a gentleman of the bedchamber. And Dr. Wallis further tells us that this Problem had been originally propoſed to Colonel Titus (as

(as the Colonel informed him,) by Dr. *John Pell*, the famous Algebräiſt of that time. So that it is a queſtion that has engaged the attention, and exerciſed the talents, of very eminent Mathematicians. But my reaſon for introducing it into this Collection of Tracts (which relates to the reſolution of equations, and not to the ſolution of problems, or the reduction of them to equations,) was it's connection with the biquadratick equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which has been the object of our attention and conſideration through ſo great a part of the foregoing Appendix to Dr. Halley's Tract: for this circumſtance, I ſuppoſed, would make my readers deſirous of becoming acquainted with the Problem from which it had been derived, and of underſtanding the Solution that had been given of that Problem by Dr. Wallis. This, however, I found to be no eaſy matter: for I attempted ſeveral times to underſtand Dr. Wallis's Solution of this Problem without being able to make it out, though I had each time employed ſeveral hours in the attempt, and had filled ſome pages of large ſheets of paper with the long Algebräick operations of multiplication, addition, and ſubtraction, which were neceſſary for that purpoſe. But theſe operations were ſo intricate and complicated that I repeatedly made miſtakes in attempting to perform them, and, after much tireſome labour, was obliged to begin them over again. At laſt, however, I performed them rightly, and obtained the equation which has been given us by Dr. Wallis, as the reſult of the conditions of the Problem, and as being the grand, final, equation by the reſolution of which the unknown quantities which the Problem requires us

to find, may be diſcovered. And this equation is an equation of the 12th order, or involves the 12th power of the unknown quantity which is it's root. And, as I ſhould be ſorry that my readers ſhould be forced to waſte their time and pains in making the ſame fruitleſs endeavours to perform theſe operations that I had made, I have, in this Tract, ſet down, at full length, the ſeveral compound quantities produced by every new operation in Dr. Wallis's Solution; ſo that the diligent reader, that ſhall chuſe to go through the Solution with care and exactneſs, and to perform all the neceſſary operations in it, may continually compare the ſeveral terms of his own reſults with thoſe of the reſults here ſet down, and thereby diſcover and correct any miſtakes he may have made in performing the operations, as ſoon as they ariſe. The conſideration of this Problem begins in page 187, and Dr. Wallis's Solution of it ends in page 217.

This Problem is, to find three numbers, denoted by the three letters a, b, and c, that ſhall be of ſuch magnitudes that $aa + bc$ ſhall be $= 16$, and $bb + ac$ ſhall be $= 17$, and $cc + ab$ ſhall be $= 18$. And, if l be put for 16, m for 17, and n for 18, and ee be put $= 2aa$, the final equation involving e, reſulting from the conditions of the Problem, is the equation of the 12th order, (or involving the 12th power of the unknown quantity e,) that is ſet down in page 217.

But this equation may be reduced to an equation of the eighth power, or order, by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, or the ſquare of $2l - e^2$; which

which diviſion is ſet down by Dr. Wallis in his Algebra, and is likewiſe ſet down in the preſent Volume in pages 218 and 219. And the equation reſulting from this diviſion is ſet down in page 226. And, in pages 220 and 221, the values of the three letters l, m, and n, to wit, the numbers 16, 17, and 18, are ſubſtituted in the terms of the ſaid equation inſtead of the ſaid letters themſelves, and the ſaid equation is thereby converted into the numeral equation $e^6 - 80e^5 + 1998e^4 - 14{,}937e^2 + 5000 = 0$; and this equation, by proper additions and ſubtractions of it's terms, is further changed into the equation $14{,}937e^2 - 1998e^4 + 80e^5 - e^6 = 5000$; and this laſt equation, by ſubſtituting x inſtead of ee in it's terms, becomes the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, which was ſo much the object of our attention in the Appendix to Dr. Halley's Diſcourſe.

In the equation of the 12th order, which is the reſult of Dr. Wallis's Solution of the foregoing Problem, and which is ſet down in page 217 of the preſent Volume, all the terms are placed on the left-hand ſide of the equation, and their reſult, or value, is declared to be equal to 0; which is the method in which Dr. Wallis and moſt other writers of Algebra, ſince the publication of Harriot's Treatiſe of Algebra in the year 1631, uſually range the terms of their equations. And this equation is afterwards reduced to an equation of the eighth order by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, which is an exact diviſor of it, or which divides it ſo as to produce a quotient conſiſting of a certain number of terms without any remainder, one of which terms is e^8;

and this quotient is alſo equal to 0. Thus we have a dividend that is equal to 0 divided by the trinomial quantity $e^4 - 4le^2 + 4l^2$, and producing a quotient that is alſo equal to 0. All this ſeems very obſcure and myſterious, and bordering upon nonſenſe. And therefore, to avoid the difficulties ariſing from this method of reducing the ſaid equation of the 12th order to an equation of the eighth order, I have, in art. 15, pages 222, 223, 224, &c, - - - - - 227, given another method of doing the ſame thing, which is perfectly clear and intelligible. For, by adding ſome quantities to both ſides of Dr. Wallis's final equation, ſet down in page 217, of which the right-hand ſide was 0, I obtain two finite compound quantities that are equal to each other; and I then divide theſe two equal quantities by the trinomial quantity $e^4 - 4le^2 + 4l^2$, and thereby obtain two ſeparate quotients, which will be equal to each other. And from this equation between theſe two quotients we may afterwards derive (by a proper arrangement of it's terms, and by ſubſtituting in it, inſtead of the letters l, m, and n, the values of thoſe letters, to wit, the numbers 16, 17, and 18,) firſt, the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, and, afterwards, (by ſubſtituting x inſtead of ee in this laſt equation) the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, which was reſolved in the foregoing Appendix to Dr. Halley's Diſcourſe. This article, I preſume, will be thought an intereſting one by all ſuch readers as are deſirous of preſerving clear ideas in all the operations that they have occaſion to perform in Algebra.

In pages 228, 229, 230, &c. ... 232, I have applied the number 12,756,441,794,480,744, or the second value of x in the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, or of v in the equation $14{,}937v^4 - 1998v^6 + 80v^8 - v^{10} = 5000$, to the Solution of Colonel Titus's Problem, by taking $aa = \frac{w}{2} \left(= \frac{12{,}756{,}441{,}794{,}480{,}744}{2}\right) =$ 6.378,220,897,240,372, and consequently $a = \sqrt{6.378{,}220{,}897{,}240{,}372} = 2.525{,}513{,}986$, or, nearly, 2.525,514; which is therefore the value of the first of the three numbers that were required to be found by that Problem. This first number a being thus found, the second number b may be derived from it by computing the expression $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, as is shewn in the course of Dr. Wallis's Solution of the Problem; and the third number c may be derived from the two former numbers a and b by computing the expression $\frac{l - aa}{b}$, as is likewise shewn in Dr. Wallis's Solution. These computations are made in pages 229 and 230, and it there appears that the expression $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ is $= 2.969{,}152{,}826$, or, nearly, 2.969,153, and that the expression $\frac{l - aa}{b}$ is $=$ 3.240,580. Therefore the three numbers sought are, nearly, the three mixt numbers 2.525,514, 2.969,153, and 3.240,580. Q. E. I.

And in art. 17 and 18, pages 230, 231, and 232, it is ſhewn that theſe three numbers will anſwer the conditions of the Problem.

But Dr. Wallis has obſerved that the firſt, or leaſt, value of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 e^8 = 5000$ will alſo enable us to find three other numbers that will anſwer the conditions of the Problem, as well as the three numbers 2.525,514, 2.969,153, and 3.240,580. And this is ſhewn in art. 19 of the preſent Tract, pages 232, 233, &c, - - 256. For this firſt, or leaſt, value of ee is $= 0.350{,}987{,}045{,}866$, &c as has been ſhewn in the foregoing Appendix. And, if we take $ee = 0.350{,}987{,}045{,}866$, &c, and $aa = \frac{ee}{2}$ $(= \frac{0.350{,}987{,}045{,}866, \&c}{2}) = 0.175{,}493{,}522{,}933$, &c, and conſequently a $(= \sqrt{0.175{,}493{,}522{,}933}$, &c) $= 0.418{,}919{,}470{,}701$, &c, and from this value of a deduce the values of b and c by computing the expreſſions $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lma^2 + m^2a^2 + 4l^2n}}{2n}$ and $\frac{l - aa}{b}$, we ſhall find the former of theſe expreſſions, or the value of the ſecond number b, to be $= 3.912{,}226{,}866$, and the latter of them, or the value of the third number c, to be $= 4.044{,}884{,}670$. And conſequently the three numbers ſought will be the decimal fraction 0.418,919,470, &c, and the two mixt numbers 3.912,226,866, and 4.044,884,670. This is ſhewn in art. 19, pages 232, 233, and 234; and in art. 20, pages 234, 235, and 236, it is ſhewn

shewn that these three numbers will answer the conditions of the Problem.

After thus discovering that two of the values of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5020$ serve equally well to enable us to find three numbers that will answer the conditions of Colonel Titus's Problem, it may be suspected that the other two values of ee in the same equation, to wit, 32.060,290,8 and 34.832,280,2, would likewise enable us to find two other sets of numbers that would also answer the conditions of the Problem. But this is found, upon trial, not to be true. For, if ee is taken $= 32.060,290,8$, and aa is taken $= \frac{ee}{2}$ $(= \frac{32.060,290,8}{2}) = 16.030,145,4$, and a is taken $= \sqrt{16.030,145,4}$, and b is taken $= \frac{ma}{2n} + \frac{\sqrt{8nn^4 - 12lnn^2 + m^2n^2 + 4l^2n}}{2n}$, or to $\frac{17a}{36} + \frac{\sqrt{144a^2 - 3167a^2 + 18,432}}{36}$, and c is taken $= \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, it will be found that the values of $aa + bc$, $bb + ac$, and $cc + ab$, resulting from these values of a, b, and c, will not be equal to the numbers 16, 17, and 18 respectively; but that $aa - bc$ (instead of $aa + bc$) will be $= 16$, and that $bb + ac$ will be (as before) $= 17$, and that $cc - ab$ (instead of $cc + ab$) will be $= 18$; and consequently that the three values of a, b, and c so obtained will be fitted to answer the conditions of a problem somewhat different from that of Colonel Titus, to wit,

wit, a problem in which it ſhould be required to find the values of three unknown quantities a, b, and c, that ſhould be of ſuch magnitudes that $aa - bc$ (inſtead of $aa + bc$,) ſhould be equal to 16, and that $bb + ac$ ſhould (as before,) be equal to 17; and that $cc - ab$ (inſtead of $cc + ab$,) ſhould be equal to 18. And, if ee is taken $= 34.832,280,2$; and aa is taken $= \frac{ee}{2}$ $(= \frac{34.832,280,2}{2}) = 17.416,140,1$, and b is taken $= \frac{17a}{36} - \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and c is taken $= \frac{1 - aa}{b}$, or $\frac{16 - aa}{b}$, it will be found that the values of $aa + bc$, $bb + ac$, and $cc + ab$, reſulting from theſe values of a, b, and c, will not be equal to the numbers 16, 17, and 18 reſpectively, but that $aa - bc$ (inſtead of $aa + bc$) will be $= 16$, and that $bc - ac$ (inſtead of $bc + ac$,) will be $= 17$, and that $cc + ab$ will be (as before) $= 18$; and conſequently that the three values of a, b, and c ſo obtained will be fitted to anſwer the conditions of a problem ſomewhat different both from Colonel Titus's Problem and from the other problem juſt now mentioned, to wit, a problem in which it ſhould be required to find the values of three unknown quantities a, b, and c, that ſhould be of ſuch magnitudes that $aa - bc$ (inſtead of $aa + bc$) ſhould be equal to 16, and that $bc - ac$ (inſtead of $bc + ac$) ſhould be equal to 17, and that $cc + ab$ ſhould (as before) be equal to 18. And thus it appears that the equation $14,937ee - 1998e^4 + 80e^6 - e^8 = 5000$ is related to, or may be derived from, three different problems that bear a reſemblance to each other

other by agreeing in ſome of their conditions, but not in all, and that two of the values of *ee* in this equation, to wit, the leaſt and the leaſt but one, are related to the firſt of the ſaid three problems, to wit, that propoſed by Colonel Titus, and will enable us to find two ſets of numbers that will anſwer the conditions of that problem; and that the third value of *ee*, in this equation, to wit, 32.060,290,8, will enable us to find three numbers that will anſwer the conditions of the ſecond problem; and that the fourth, or greateſt, value of *ee* in the ſame equation, to wit, 34.832,280,2, will enable us to find three numbers that will anſwer the conditions of the third problem. This is a remarkable concluſion, and deſerves to be attended to. Theſe obſervations are briefly mentioned in art. 21, pages 236, 237, and 238; but, for more ample ſatisfaction concerning theſe two greateſt values of *ee* in the ſaid equation, the reader is referred to Dr. Wallis's Algebra, Chapter LXII, art. 63, 64, 65, 66, &c, - - 79, where he will find the ſubject fully diſcuſſed, but not, it is apprehended, without ſome degree of obſcurity ariſing, partly, from the conſideration of negative quantities, and, partly, from the doctrine of the generation of equations, one from another, by multiplication, or by bringing all the terms of the equation to the firſt, or left-hand, ſide of the equation, ſo as to make them equal to 0, and then multiplying the equations (ſo prepared and made equal to nothing,) one into another; and from the eminently-falſe poſition derived from that manner of generating equations, to wit, "that every Algebraick equation has as many roots as there are units in the index of the higheſt power of the unknown quan-

tity contained in the equation." For this doctrine of the generation of equations, one from another, by multiplication, (which was invented by Harriot, and adopted by Des Cartes, and Dr. Wallis, and almoſt all the ſubſequent writers on Algebra,) inſtead of being an improvement in that ſcience (as many people conſider it), has, in my opinion, been of great detriment to it by deſtroying it's ſimplicity and perſpicuity, and therefore ought again to be diſcarded from it.

The foregoing Solution of Colonel Titus's Problem, given us by Dr. Wallis, is (as we have already obſerved,) exceedingly tedious and laborious: and great part of the labour required in the ſaid Solution ariſes from the neceſſity we are under, of raiſing the equation which involves the unknown quantity a to the 12th degree, or to the 12th power of a, in order to free it's terms from radicality. But Mr. William Frend, the ingenious author of the late perſpicuous Treatiſe on Algebra in one volume, octavo, intitled, *Principles of Algebra*, (in which he totally rejects the abſurd and perplexing doctrine of *negative quantities*, or *quantities leſs than nothing*, or *quantities obtained by the ſubtraction of a greater quantity from a leſſer*,) has lately communicated to me another Solution of this Problem which produces only a biquadratick equation, to wit, the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. And this equation is found to have three roots, or, in the language of modern Algebraiſts, three *real and affirmative* roots; of which the middle root will enable us to find the firſt ſet of values of the three unknown numbers a, b, and c, that will anſwer the conditions

conditions of Colonel Titus's Problem, to wit, the three numbers 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c; and the greateſt root will enable us to find the ſecond ſet of values of the ſaid three unknown numbers, that will anſwer the ſame conditions, to wit, the three numbers 0.418,919,47, 3.912,226,8 &c, and 4.044,884,6 &c. This Solution of this Problem I have therefore inſerted in this Collection, immediately after the foregoing Solution of it by Dr. Wallis, and have exhibited and explained it very fully. It begins in page 240, and ends in page 275.

In this Solution Mr. Frend puts $b = xa$, and $c = ya$; and in the courſe of his Solution ſhews that y will be $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$. So that, when x is diſcovered, the proportion of c, or ya, to a will be known, as well as the proportion of b, or xa, to a. His object in the Solution is therefore to find x. This Solution of Mr. Frend is contained in art. 23, 24, 25, and 26, and pages 240, 241, 242, 243, and 244, and produces the biquadratick equation $x^4 - \left(\frac{ln + m^2}{mn - l^2}\right) \times x^3 + \frac{4lm}{mn - l^2} \times xx - \left(\frac{l^2 + mn}{mn - l^2}\right) \times x = \frac{m^2 - ln}{mn - l^2}$. And in art. 27, pages 244 and 245, it is ſhewn that, if we ſubſtitute in this equation, inſtead of the letters l, m, and n, the numbers that are denoted by them, to wit, 16, 17, and 18, the ſaid equation will be converted into the numeral equation $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x =$

$x = 0.02$. This equation muſt therefore be reſolved, in order to obtain the value of x; and from the value of x ſo obtained, or, if the equation ſhould have more than one root, (as is the caſe,) from that value of x which has a relation to the preſent Problem, we muſt then derive the value of y by computing the fraction $\frac{lx^2 - m}{nx - l}$ or $\frac{16x^2 - 17}{17x - 16}$, to which y is equal. And, when this is done, we muſt multiply the value of xa, or b, into the value of ya, or c; which will give us the value of bc expreſſed by it's relation to aa: and then, by ſubſtituting this value of bc inſtead of bc in the equation $aa + bc = 16$, we ſhall convert the ſaid equation into a quadratick equation involving only one unknown quantity, namely, a, and which may therefore be eaſily reſolved. And the reſolution of this quadratick equation will give us the value of a, or the firſt of the three unknown quantities a, b, and c, that the Problem requires us to find. And from the value of a, ſo found, we may derive the values of b and c by multiplying a into x to obtain the number b, and by multiplying it into y to obtain the number c. Our chief buſineſs therefore is to find the roots of the biquadratick equation $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x = 0.02$, which is done very fully in pages 246, 247, 248, &c, - - 273, by the application of Mr. Raphſon's method of approximation; and the firſt, or leaſt, root of this equation is found (in pages 246, 247, 248, &c, - - 255,) to be very nearly $= 1.027,179,787$, which, it is ſhewn in art. 34, page 235, will not enable us to ſolve Colonel Titus's

Titus's Problem; and the ſecond, or middle, root is found (in art. 35, 36, 37, &c, - - - 40, pages 256, 257, &c, - - - 265) to be nearly = 1.175,056,46; and this root is (in art. 41 and 42, pages 266 and 267,) applied to the Solution of Colonel Titus's Problem, and is found to anſwer the purpoſe, by enabling us to find 2.525,534 for the value of a, or the firſt of the three unknown numbers a, b, and c, which we were required to diſcover, and 2.969,159 for the value of (xa, or) b, or the ſecond of the ſaid three numbers, and 3.240,537 for the value of (ya, or) c, or the third, or laſt, of the ſaid numbers; which three values of a, b, and c are nearly the ſame with the values of them found before in the firſt part of Dr. Wallis's Solution of this Problem; and, laſtly, the third, or greateſt, root is found (in art. 44, 45, 46, pages 268, 269, &c, - - - 273) to be nearly = 9.338,851,9; and this root is (in art. 47 and 48, pages 273, 274, and 275,) applied to the Solution of Colonel Titus's Problem, and is found to anſwer the purpoſe, by enabling us to find 0.418,919,47 for the value of a, and 3.912,226,888 for the value of (xa, or) b, and 4.044,884,650 for the value of (ya, or) c; which three values of a, b, and c are nearly the ſame with the latter three values of the ſame quantities found before in the ſecond part of Dr. Wallis's Solution of this Problem.

This Problem is therefore now compleatly ſolved by Mr. Frend's method of proceeding, as it had been before by that of Dr. Wallis. And Mr. Frend's Solution has this advantage over that of Dr. Wallis, that it ſaves us

the trouble of thoſe very tedious and perplexing Algebräical multiplications, ſubtractions, and diviſions, which were neceſſary in Dr. Wallis's Solution, and which it is very difficult to perform without making ſome ſlip, or error, either in ſetting down the ſigns + and —, that are to be prefixed to the ſeveral terms, or in computing the co-efficients of the terms, or the indexes of the powers of the quantities involved in them.

The inveſtigations of the three roots of the biquadratick equation $x^4 - 11.54 \times x^3 + 21.76x^2 - 11.24 \times x = 0.02$ are eminent inſtances of the utility of Mr. Raphſon's method of reſolving equations by approximation, and in that view, as well as on account of their relation to Colonel Titus's Problem, were fit to receive a place in the preſent Volume, which is intended to illuſtrate and recommend that method of reſolving equations.

The next Tract in the preſent Volume is re-printed from my Appendix to Mr. Frend's *Principles of Algebra*; and it had been publiſhed for the firſt time in the large octavo volume of Tracts containing a part of Mr. James Bernoulli's excellent Treatiſe *De Arte Conjectandi*, and other mathematical pieces, in the year 1795. It is intitled, *Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all Degrees by Approximation.* It begins in page 279, and ends in page 323, and it's contents may be deſcribed as follows. The firſt part of it, as far as page 303, is intended, partly, to remove ſome difficulties that occur in reading Mr. Raphſon's excellent Treatiſe

Treatiſe on the Reſolution of all Equations, (whether pure or affected,) by Approximation, intitled *Analyſis Æquationum Univerſalis*; which difficulties are not inherent in the ſubject itſelf, or neceſſarily belonging to his method of reſolving equations, but have ariſen merely from his having unfortunately adopted the doctrine and language of *negative roots* of equations, by which the Science of Algebra, or Univerſal Arithmetick, has been diſgraced and rendered obſcure and difficult, and diſguſting to men of a juſt taſte for accurate reaſoning, ever ſince it's introduction by Harriot and Des Cartes. The firſt part of this Tract is, I ſay, intended, partly, to remove ſome difficulties of this kind in the ſaid Treatiſe of Mr. Raphſon, and, partly, to illuſtrate his method of reſolving high equations in other caſes, or where no negative roots are mentioned, by performing the reſolution of one of the equations adduced by him as examples of his method, to wit, the reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, in a very full and diſtinct manner, with every ſtep of the reſolution, and the reaſonings upon which it is grounded, ſet forth at length, agreeably to the principles laid down by him in the beginning of the ſaid Treatiſe, inſtead of reſorting (as he has done in his reſolution of the ſame equation, and in thoſe of all the other equations adduced by him as examples of his method,) to the repeated application of a general theorem, or canon, which he has deduced from the ſaid principles: becauſe I was of opinion that the latter way of performing the ſaid reſolution, by means of a theorem, or canon, affords much leſs ſatisfaction to the mind of the reader, or calculator,

in the uſe of it than he would receive by reſolving the equation in the former way, or by the immediate application of the principles themſelves, as is done in the reſolution here given of the ſaid equation. And the following part of this Tract (from page 303 to page 317,) contains ſome obſervations on the reſemblance between Mr. Raphſon's method of reſolving an equation by approximation and Sir Iſaac Newton's method of doing the ſame thing; which latter method in it's firſt proceſs, (or in finding the ſecond near value of x, or the root of the equation, after having found a, or it's firſt near value, by conjecture, or otherwiſe, to a moderate degree of exactneſs,) is preciſely the ſame with Mr. Raphſon's, but in the following proceſſes differs a little from it. This circumſtance makes it neceſſary to compare the two methods with each other, by applying them to the reſolution of the ſame equation; as we before compared Mr. Raphſon's method with Dr. Halley's, by applying them both ſucceſſively to the reſolution of the ſame three equations in the foregoing Appendix to Dr. Halley's Diſcourſe. And with this view I have (in pages 305, 306, 307, - - - 317) reſolved the cubick equation $x^3 - 2x = 5$ (which is the equation adduced by Sir Iſaac Newton as an example of his method,) by both Mr. Raphſon's method and Sir Iſaac Newton's method, in a very full and diſtinct manner, and ſo as to ſet the reſpective advantages of the two methods in oppoſition to each other, to the end that the reader may be enabled to compare them one with the other, and determine which of the two is, upon the whole, the more convenient. And here I muſt confeſs that it appears to me,

upon

upon making this compariſon, that Mr. Raphſon's method of approximation is, in ſome degree, preferable even to Sir Iſaac Newton's method, though not ſo much as to that of Dr. Halley.

In art. 22, pages 317, 318, and 319, a proof is given of the exactneſs of the number 2.094,551,48, which has been found by both Mr. Raphſon's and Sir Iſaac Newton's methods of approximation to be the root of the equation $x^3 - 2x = 5$. And, in art. 23, pages 321, 322, and 323, there are ſome obſervations on the difficulty of finding a, or the firſt near value of an affected equation, in certain caſes, to wit, in thoſe caſes in which the propoſed equation either has, or (from the changes of the ſigns of it's terms from + to − and from − to +,) ſeems to have, more than one real and affirmative root. And with theſe obſervations this Tract concludes.

The next Tract in this Collection is the 10th Chapter of the 2nd Book of Mr. John Kerſey's *Elements of Algebra*,—a moſt valuable work in two ſmall volumes, folio, that was publiſhed in the year 1673. This book is now too little known, and deſerves to be ſtrongly recommended to all ſtudents of Algebra on account of the great fulneſs and perſpicuity with which it is written; in which important qualities it very far ſurpaſſes moſt of the Treatiſes of Algebra that have been written ſince, excepting Dr. Saunderſon's *Elements of Algebra*, in two volumes, quarto, which was publiſhed in the year 1740. And in ſome reſpects it is even preferable to Saunderſon's Algebra; becauſe it gives a fuller account of the Diophantine problems, and becauſe it deals leſs in the

abſurd myſteries of *negative roots*, and *negative quantities* in general, and the doctrine of the generation of higher affected equations from lower equations by multiplication, which has been the ſource of great obſcurity and perplexity to all the ſtudents of Algebra ever ſince it's introduction into that ſcience by the publication of Harriot's Algebra in the year 1631 and Des Cartes's Geometry in the year 1637. Theſe abſurd doctrines ſeem to have delighted Dr. Saunderſon, who frequently inſults and rails at ſuch of his readers as ſhall find a difficulty in underſtanding them, as being perſons, of narrow minds and incorrigible dullneſs and ſtupidity: but Mr. Kerſey, though he did not venture boldly to reject them, as either poſitively abſurd and unintelligible or, at leaſt, leading to obſcurity, yet ſays but little about them, and ſeems deſirous, as much as poſſible, to keep clear of them. His book therefore deſerves to be ſtudied by all perſons who deſire to become ſkilfull in Algebra; and to confirm this recommendation of it I will add the following paſſage from the Preface of the learned Dr. Wallis to the Latin Edition of his Algebra, in the ſecond volume of the Collection of all his Works publiſhed at Oxford in three volumes, folio, in the year 1693, where, after mentioning other eminent writers on this ſubject, he has theſe words: *Suaſerim ut lector conſulat ex noſtris (præter alios,) Kerſæum noſtrum; qui duobus voluminibus integram Algebræ tractationem exhibuit, fusè quidem et perſpicuè traditam: quo nemo felicius quæſtiones Diophantæas elucidavit.* The chapter of this valuable book which is here reprinted, (and which I had reprinted already about a year ago in the Appendix to Mr. Frend's *Principles of Algebra*,) relates to the method invented by Simon

Stevinus, the Flemiſh Mathematician, (who flouriſhed in the beginning of the laſt century, and died in the year 1633,) for finding a, or the firſt near value of x, or the root of any propoſed Algebräick equation, by repeated conjectures and trials with eaſy numbers of one or two decimal figures: after having found which we may proceed to determine the value of the ſaid root to a greater degree of exactneſs by one or more applications of Mr. Raphſon's method of approximation. The title of this 10th Chapter of Mr. Kerſey's Algebra is as follows: *An Explication of Simon Stevin's General Rule to extract one Root out of any poſſible Equation in Numbers, either exactly or very nearly true.*

I have next inſerted in this Collection of Tracts a Remark, (which, as well as the two preceeding Tracts, I had printed before in my Appendix to Mr. Frend's *Principles of Algebra*,) on a groſs error in the reaſoning of the late celebrated French Mathematician, Monſieur Clairaut, in his elegant little treatiſe, intitled *Eléments d'Algèbre*, where he endeavours to prove that, if two negative quantities, called $-b$ and $-d$, be multiplied together, their product will be $+bd$. This Propoſition he ſhould have, firſt, made intelligible, by telling us what he meant by a negative quantity, before he undertook to demonſtrate it. But this he has not done. For, in the beginning of his book, he ſpeaks of the ſign —, or *minus*, as being only the ſign of the ſubtraction of the quantity to which it is prefixed from another quantity which is greater than it; according to which meaning of the ſign —, (which is very clear and juſt) the compound quantity $a - b$ muſt ſignify the exceſs of

the quantity a above the quantity b, which is ſuppoſed to be leſs than a. He then proves clearly and by juſt reaſonings that, if the compound quantity $a - b$ is multiplied into the compound quantity $c - d$, the product of the ſaid multiplication will be the quadrinomial quantity $ac - bc - ad + ld$. But then he ſuppoſes a and c to become equal to o, while b and d retain their former magnitudes, not conſidering that, according to the only idea which he had before given us of the ſign —, the quantity a muſt be greater than b, and the quantity c muſt be greater than d, in order to make it poſſible for the compound quantities $a - b$ and $c - d$ to exiſt; and then he concludes that, ſince a and c are become equal to o, the three terms ac, $- bc$, and $- ad$, (which involve them) will become equal to o likewiſe, and the whole quadrinomial quantity $ac - bc - ad + bd$ will become equal to it's laſt term $+ bd$, and conſequently that $\overline{o - b} \times \overline{o - d}$, or $- b \times - d$, will be equal to $+ bd$. This is very weak reaſoning for ſo great a Mathematician as Monſieur Clairaut: and, if he, who was ſo juſtly celebrated for the clearneſs of his ſtyle and the accuracy of his reaſonings on many occaſions, could offer nothing better than this fallacious argument in ſupport of this favourite doctrine of negative quantities, or quantities leſs than nothing, or quantities ariſing from the ſubtraction of ſome quantities from others that are leſs than themſelves (which are the definitions given of theſe quantities by Sir Iſaac Newton, Dr. Saunderſon, and Mr. Mac Laurin, as is ſhewn in the Note in page 286 of the preſent Volume,) I think we may fairly conclude that this doctrine cannot be defended:

Si Pergama dextrâ
Defendi poſſent, etiàm hâc defenſa fuiſſent.

This Remark begins in page 339, and ends in page 343.

The next Tract in this Collection is intitled, *A General Method of inveſtigating the two, or three, firſt Figures of the leaſt Root of an Equation that has more than one real and affirmative Root*; and is reprinted from the 3d Volume of the *Scriptores Logarithmici*, pages 725, 726, &c, - - - - 761.

The foregoing method of finding *a*, or the firſt near value of the root of a propoſed equation, by means of conjectures and trials with eaſy numbers of only one or two figures, in the manner recommended by Stevinus and Kerſey, (which is ſet forth in the 10th chapter re-printed from Kerſey's Algebra in pages 325, 326, &c, - - - 336 of this Volume,) will be found perfectly convenient in all equations that have but one root. But, when the propoſed equation has, or (from the changes of the ſigns + and —, prefixed to it's terms, from + to — and from — to +,) ſeems to have, two, or three, or four, or more, roots, or (in the language of modern Algebraiſts,) real and affirmative roots, we may ſometimes be at a loſs to know which of the ſaid ſeveral roots we ought to go firſt in purſuit of. In theſe caſes therefore I am inclined to think that it will be moſt prudent, in general, to begin with the inveſtigation of the leaſt root. And the manner of doing this is the ſubject of this Tract. The propoſition on which this method of proceeding

proceeding is founded, is as follows: to wit, "that, if all the terms of the propoſed equation that have the ſign — prefixed to them are expunged from it, the root of the equation ſo curtailed will be leſs than the leaſt root of the original equation;" whence it follows that, if this root is found to two places of figures by Stevinus's method of conjecture and trial, or otherwiſe, and be called a, it will ſerve for the ground-work of an approximation to the value of the leaſt root of the original equation in Mr. Raphſon's method by putting the ſaid leaſt root equal to $a + z$, and ſubſtituting $a + z$ inſtead of x in the terms of the original equation, and reſolving the tranſformed equation reſulting from this ſubſtitution, as if it were a mere ſimple equation, in the manner recommended by Mr. Raphſon. The principle, or propoſition above-mentioned, upon which this method is grounded, is deſcribed and proved in the Lemma ſet forth and demonſtrated in pages 361, 362, 363, &c, - - - 367; and ſome examples are given of it in pages 368, 369, &c, - - - 431, which are reſolved at great length and with great care and labour: and methods are pointed-out by which, when we have obtained the leaſt, or, perhaps, the only, root of the propoſed equation, we may, in the firſt place, diſcover whether the ſaid equation has, or has not, any other root beſides that which has been already found, and, ſecondly, if the ſaid equation has any other root, we may reduce the equation to another equation of the next lower order, or degree, of which the roots will be the ſame with the remaining roots of the original equation.

After having dwelt ſo long upon Dr. Halley's, Mr. Raphſon's, and Sir Iſaac Newton's, methods of reſolving equations

equations by approximation, I conceived that it would be agreeable to the readers of theſe Tracts to be made acquainted with Vieta's method of reſolving them alſo by approximation, which was invented by that celebrated author above a hundred years before the publication of Mr. Raphſon's and Dr. Halley's methods, and which was generally adopted and practiſed by the ableſt Mathematicians of thoſe times, and particularly by Mr. Oughtred, Dr. Pell, and Dr. Wallis, though it has ſince been abandoned on account of the greater facility and expedition of the methods of Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley. I have therefore added another Tract to thoſe already mentioned, in which I have endeavoured to explain the principles upon which Vieta's method of reſolving equations by approximation is grounded, and the manner in which we are enabled by it to inveſtigate the value of the root of any equation whatſoever. I have, however, confined myſelf, in this attempt, to the reſolution of only one affected equation by this method, as a ſpecimen of it, to wit, the equation $x^3 - 5x^2 + 500x = 7{,}905{,}504$, which is the higheſt and moſt complicated of all the affected equations which Vieta has adduced as examples of his method of reſolution, and which is reſolved in the 15th Problem of his Diſcourſe on this ſubject, which he intitles *De Numeroſa Poteſtatum Adfectarum Reſolutione.* But, as Vieta's method of reſolving *affected* equations by approximation is ſimilar to, and derived from, his method of reſolving *pure* equations of the ſame degree, or of extracting the roots of numbers, I have begun my explanation of his method of reſolving the affected equation $x^3 - 5x^2 + 500x = 7{,}905{,}504$ by, firſt, ſhewing how

how he would reſolve the pure equation $x^5 = 7,905,504$, or extract the fifth root of the number 7,905,504. This is done very fully and diſtinctly in art. 4, 5, 6, and 7, pages 441, 442, &c, - - - - 451; and the reſult is, that the above fifth root is = 23.96 &c. I then (in art. 9, pages 452, 453) compare this reſult with the value of the ſame fifth root obtained by means of Monſieur de Lagny's general expreſſion for the *m*th root of any given number N, upon a ſuppoſition that *a*, or a firſt near value of it, is already known. This expreſſion has been given in the Note in page 9 of this Volume, and is $a + 2a \times \frac{N - a^m}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or, when *m* is = 5, (as it is in the preſent caſe,) $a + 2a \times \frac{N - a^5}{4N + 6a^5}$, or $a + a \times \frac{N - a^5}{2N + 3a^5}$; which, if *a* is = 23.9, will be (= 23.9 + 0.065,466,9, or) 23.965,466,9; of which number the firſt eight figures, 23.965,466, are, probably, exact: So that no leſs than five new figures, to wit, the figures 0.065,466, are obtained exactly by means of this expreſſion of Monſieur de Lagny, from the firſt three figures, 23.9, that were already known; whereas by Vieta's method of proceeding only one new figure, to wit, the figure 0.06, was obtained by the correſponding proceſs from the ſame three figures, 23.9, that were already known. I then compare the ſame reſult of Vieta's method of extraction, to wit, the number 23.96, with the value of the ſame fifth root of the ſame number 7,905,504 obtained by means of Mr. Raphſon's ſimpler expreſſion for the *m*th root of any given number N, upon a ſuppoſition that *a*, or a firſt near value of it, is already known,

known, to wit, the expreſſion $a + \frac{N - a^m}{ma^{m-1}}$, or, when m is $= 5$, (as it is in the preſent caſe,) $a + \frac{N - a^5}{5a^4}$; and I find that, if a is ſuppoſed to be $= 23.9$, this expreſſion will be $=$ (23.9 + 0.065,8, or) 23.965,8, which agrees with the more accurate value of the ſame fifth root found before by Monſieur de Lagny's expreſſion, to wit, 23.965,466,9, in the five firſt figures 23.965, which therefore muſt be exact: ſo that even this ſimple expreſſion of Mr. Raphſon has given us two new figures of the fifth root ſought, to wit, the two figures 0.065, exactly, in addition to the firſt three figures 23.9, which were already known; whereas by Vieta's method of proceeding only one new figure, to wit, the figure 0.06, was obtained by the correſponding proceſs in addition to the ſame three figures 23.9, that were already known.

By theſe compariſons it appears that both Mr. Raphſon's and Monſieur de Lagny's expreſſions for the value of the fifth root of the given number 7,905,504 have greatly the advantage of Vieta's method of extracting the ſame root in point of expedition, though not in point of accuracy of reaſoning and certainty of the concluſions obtained by them; in which reſpects they are all equally valuable.

In art. 11, pages 454, 455, and 456, I have pointed out the reſemblance between Vieta's method of extracting the fifth root of the number 7,905,504, and Mr. Raphſon's method of extracting it, and ſhewn that they

are

are almoſt the ſame. For they both proceed by a diviſion of the ſame number, or dividend, 107,391.348,01, by the ſame diviſor 1,631,404.3205; which occurs firſt in page 449, in extracting the fifth root of 7,905,504 by Vieta's method, and afterwards in page 454 in finding the ſame fifth root by computing Mr. Raphſon's expreſſion $a + \frac{N - a^m}{ma^{m-1}}$: and the only difference between them is that Mr. Raphſon continues the diviſion to three figures in the quotient, to wit, the three figures 0.0658, and thereby obtains 23.9 + 0.0658, or 23.9658, for the next near value of the fifth root ſought, of which number all the figures, except the laſt figure 8, are exact; whereas Vieta carries the diviſion to only one figure in the quotient, to wit, the figure 0.06, and thereby obtains only 23.9 + 0.06, or 23.96, for the next near value of the fifth root ſought. If Vieta had happened to obſerve that the two firſt figures of the quotient 0.0658 would have been exact, and that, in general, if n is the number of figures that are exact in a, or the part of the root that is already known, the firſt $n - 1$ figures of the quotient would be exact, and had accordingly carried his diviſion to that number of figures in the quotient, his method of extracting the fifth, or the mth, root of any given number N would have been exactly the ſame with Mr. Raphſon's.

Having thus gone through the extraction of the fifth root of the number 7,905,504, or the reſolution of the pure equation $x^5 = 7,905,504$, in the manner in which Vieta had directed ſuch reſolutions of pure equations to be

be performed, I proceed (in art. 12, 13, 14, and 15, pages 456, 457, 458, &c, - - - 463,) to exhibit Vieta's method of reſolving the affected equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$, which is ſimilar to the foregoing method of reſolving the pure equation $x^5 = 7{,}905{,}504$. And by this reſolution it appears that the root x of that equation is exactly $= 24$. I then (in art. 16, page 463,) reſolve the ſame equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ by Mr. Raphſon's method of approximation, in order to compare the proceſſes of this reſolution with thoſe in Vieta's reſolution juſt before exhibited. And I find that there is the ſame reſemblance between theſe two methods of reſolving this affected equation that I had before obſerved to take place between the ſame two methods of reſolving the pure equation $x^5 = 7{,}905{,}504$. For here, as before, both methods of reſolution lead us to the diviſion of the ſame dividend, to wit, the number 4,735,504, by the ſame diviſor, 794,500. But Vieta carries the diviſion to only one figure in the quotient, which in the preſent example is ſufficient for his purpoſe, becauſe this equation has the whole number 24, conſiſting of only two figures, for it's exact root, but which, if the root were an irrational quantity, or incommenſurable to unity, (as it moſt commonly is,) and were required to be found to ten, or twelve, places of figures, would make his method of reſolution exceſſively tedious and laborious; whereas Mr. Raphſon continues his diviſions to as many figures in the quotients, or to as many wanting one, as there are figures that are exact in a, or the part of the root x that is already known; by which means, if the figures that are exact in a, or the known

part

part of the root, are five, or ſix, or ſeven, he will obtain four, or five, or ſix, new figures in the quotient to be added to a, that will be all exact. And in this circumſtance alone conſiſts the advantage of Mr. Raphſon's method of approximation over that of Vieta. If Vieta had happened to obſerve that theſe diviſions might be ſafely continued to ſeveral figures in the quotients, or that ſeveral figures in the quotients, (namely, as many, wanting one, as there are figures that are exact in a, or the part of the value of x that is already known,) would be exact, and had conſequently directed his readers to continue the diviſions to that number of figures in the quotients, his method would, as I conceive, have been the very ſame with Mr. Raphſon's, and the art of reſolving all ſorts of equations by approximation would not only have been invented by him (as it has been,) but would have been brought by him at once to the higheſt degree of perfection of which it, probably, is capable. He therefore ought to be conſidered as the original founder of the whole doctrine of reſolving equations by approximation, though Sir Iſaac Newton, Mr. Raphſon, Monſieur de Lagny, and Dr. Halley, and, perhaps, ſome ſubſequent Mathematicians, have made valuable improvements on his method, by which the practice of it has been very much facilitated.

This Specimen of Vieta's Method of reſolving Equations by approximation begins in page 435, and ends in page 470.

The laſt Tract in this Collection is written by the learned and ingenious Mr. Frend, the author of a valuable

able Treatiſe of Algebra in one volume, octavo, intitled, *Principles of Algebra*, in which he has totally diſcarded the obſcure and myſterious doctrines of negative quantities, and impoſſible, or imaginary, roots of equations, and the generation of higher equations from lower ones by bringing all the terms of the lower equations to the ſame ſide of the equation, ſo as to make the reſults of them equal to nothing, and then multiplying them, (when thus made equal to o,) into each other, and thereby producing equations of higher orders that will likewiſe be equal to nothing. His preſent Tract, which is intitled "*Remarks on the Number of Negative and Impoſſible Roots in Algebraick Equations*," has the ſame laudable tendency to free Algebra from obſcurity, by ſhewing more clearly than is uſually done by writers on this ſubject, what is meant by *the negative roots* of an equation, to wit, "that they are in truth the roots, or (in the language of modern Algebraiſts,) the *affirmative*, or *poſitive*, roots, of another equation which is derived from the firſt equation by changing the ſigns + and — that are prefixed to ſuch of it's terms as involve any odd powers of the unknown quantity x, into the contrary ſigns," and by obſerving that *impoſſible roots* are mere fictitious quantities, of which no clear idea can be formed, which have been introduced into Algebra to make-up the number of roots which *Harriot* and *Des Cartes* and their followers have thought proper to aſcribe to every equation of any given degree, or order, denoted by any whole number called m, to wit, m roots; though, in truth, there is only one form in every degree, or order, of equations, (as, for example, the form $px - x^2 = q$ in quadratick equations,

and the form $x^3 - px^2 + qx = r$ in cubick equations, and the form $rx - qx^2 + px^3 - x^4 = s$ in biquadratick equations,) that can really have ſo many roots. When therefore an equation occurs, of the mth order, that has fewer than m roots, the modern Algebräiſts firſt call-in *the negative* roots of the equation, (or *the real and affirmative* roots of the ſecond equation which is derived from the firſt equation in the manner above-deſcribed,) in aid of the former, or real and affirmative, roots of it; and, if the ſum of both the affirmative and the negative roots of the equation, (which ſum we will ſuppoſe to be denoted by the letter n,) is likewiſe leſs than m, (which is often the caſe,) they then declare that, beſides the n real roots, both affirmative and negative, the equation has $m - n$ *impoſſible*, or *imaginary*, roots, which, being added to the former n roots, makes up the full number of m roots, which they had before declared that the ſaid equation ought to have. Theſe Remarks begin in page 473, and end in page 479. And with theſe Remarks the preſent Volume concludes.

A Note

A Note on Page 242.

IN page 242, &c, Mr. Frend's Solution of Colonel Titus's Problem may be made ſhorter and eaſier by making the following changes in it.

In art. 25, lines 6 and 7, we have $myy - ny = nxx - mx$. From this equation let the Solution be as follows.

But it has been ſhewn in art. 24 that y is $= \frac{lx^2 - m}{mx - l}$.

Therefore yy will be $(= \frac{\overline{lx^2 - m}^2}{\overline{mx - l}^2}) = \frac{l^2x^4 - 2lmx^2 + m^2}{\overline{mx - l}^2}$, and myy will be $= \frac{l^2mx^4 - 2lm^2x^2 + m^3}{\overline{mx - l}^2}$; and ny will be $= \frac{lnx^2 - mn}{mx - l} = \frac{\overline{lnx^2 - mn} \times \overline{mx - l}}{\overline{mx - l}^2} = \frac{lmnx^3 - l^2nx^2 - m^2nx + lmn}{\overline{mx - l}^2}$; and conſequently $myy - ny$ will be $= \frac{l^2mx^4 - 2lm^2x^2 + m^3}{\overline{mx - l}^2}$ — the fraction $\frac{lmnx^3 - l^2nx^2 - m^2nx + lmn}{\overline{mx - l}^2} = \frac{l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn}{\overline{mx - l}^2}$.

But $myy - ny$ is $= nxx - mx$.

 Therefore

Therefore the fraction juſt now obtained, to wit, $\frac{l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn}{\overline{mx-l}^2}$ will alſo be $= nxx - mx$; and (multiplying both ſides into $\overline{mx-l}^2$,) the ſeptinomial quantity $l^2mx^4 - lmnx^3 - 2lm^2x^2 + l^2nx^2 + m^2nx + m^3 - lmn$ will be $(= \overline{nxx-mx} \times \overline{mx-l}^2 = \overline{nxx-mx} \times \overline{m^2x^2-2lmx+l^2})$ = the ſextinomial quantity $m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$. Therefore (adding $lmnx^3 + 2lm^2x^2$ to both ſides,) we ſhall have $l^2mx^4 + l^2nx^2 + m^2nx + m^3 - lmn = m^2nx^4 - lmnx^3 + l^2nx^2 - m^3x^3 + 4lm^2x^2 - l^2mx$, and (ſubtracting $l^2mx^4 + l^2nx^2 + m^2nx$ from both ſides,) $m^3 - lmn = m^2nx^4 - l^2mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx$, and (dividing all the terms by m,) $m^2 - ln = mnx^4 - l^2x^4 - lnx^3 - m^2x^3 + 4lmx^2 - l^2x - mnx = \overline{mn-l^2} \times x^4 - \overline{ln+m^2} \times x^3 + 4lm \times x^2 - \overline{l^2+mn} \times x$, and (dividing all the terms by $mn - l^2$, the co-efficient of x^4, the higheſt term,) $x^4 - \frac{ln+m^2}{mn-l^2} \times x^3 + \frac{4lm}{mn-l^2} \times x^2 - \frac{l^2+mn}{mn-l^2} \times x = \frac{m^2-ln}{mn-l^2}$; which is the final equation obtained, with more difficulty, at the end of art. 26, page 244.

This improvement of Mr. Frend's Solution was ſuggeſted to me by my learned friend, the Reverend Mr. John Hellins, B. D. Vicar of Potter's Pury in Northamptonſhire.

E R-

ERRATA.

IN page 7, line 4, instead of 6.358, read 6.1358.

In page 8, line 11, after the word *had*, insert *been*.

In page 10, line 11, under the radical sign, instead of $\frac{1}{35}aa$, read $\frac{1}{36}aa$.

In page 12, line 18, after the word *column*, dele *to*.

In page 16, line 6, instead of 10.26, read — 10.26.

In page 66, line 13, instead of *consequently*, read *considerably*.

In page 114, line 2 from the bottom, instead of zaz, read $2az$.

In page 139, line 12, instead of $17d$, read $17d^2$.

And in the same page 139, line 16, instead of $17d$, read $17d^2$.

In page 161, lines 7 and 9, instead of 0.000,000,045,866,14, and 0.350,987,045,866,14, read 0.000,000,045,866,06, and 0.350,987,045,866,06. For, upon a re-examination of the computation in page 431, which makes z = 0.000,000,045,866,06, and not = 0.000,000,045,866,14, I have found it to be right.

In page 168, line 7 from the bottom, before $\overline{34.8}^2$, insert the mark =.

In page 204, line 11 from the bottom, instead of $168m^4na^8$, read $168m^4na^8$.

In page 212, line 5, instead of $4l^4mna^2$, read $4lm^7n^7a^2$.

In page 218, line 9, instead of $+ mne^8$, read $- mne^8$.

In page 227, line 14, instead of $7le^2$, read $7l^3e^2$.

And in the same page 227, line 4 from the bottom, instead of $7le^2$, read $7l^3e^2$.

In page 247, line 6, inſtead of 11.24, read 11.24x.

And in the ſame page 247, line 2 from the bottom, inſtead of 0.2, read 0.02.

In page 249, line 3, inſtead of 1.112,453,441, read 1.112,453,263,441.

In page 297, line 3 from the bottom, inſtead of xx, read x.

In page 353, line 19, inſtead of *the*, read *thoſe*.

And in the ſame page 353, lines 2 and 4 from the bottom, after the number 14,937, and after the word *increaſes*, inſert a comma.

In page 390, line 3, before the number 12,721.120, inſert the mark =.

In page 393, line 15, under the vinculum, inſtead of $ba + ba$, read $b^2a + ba^2$.

In page 404, line 2 from the bottom, after the word *equal*, inſert *to*.

In page 430, line 4 from the bottom, inſtead of —, read +.

In page 449, line 7, inſtead of $5 \times 23 \times 0.9$, read $5 \times 23 \times 0.6561$.

In page 454, line 12, inſtead of $\sqrt{5}\overline{7,905,504}$ read $\sqrt{5}\overline{7,905,504}$.

In page 455, line 15, inſtead of 1,631,404,3205 read 1,631,404.3205.

A Table

A Table of the Contents of this Collection of Mathematical Tracts.

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A New, Exact, and Eaſie Method of finding the Roots of any Equations Generally; and that without any previous Reduction.

By Dr. EDMUND HALLEY.

Being Number 210 *of the Philoſophical Tranſactions, publiſhed in May,* 1694.

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An Appendix to the foregoing Tract of Dr. Edmund Halley on the Reſolution of Algebraick Equations of all Degrees by Approximation.

By FRANCIS MASERES, Eſq. F. R. S.

CURSITOR BARON OF THE COURT OF EXCHEQUER IN ENGLAND.

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NUMBER IX.

A Specimen of Vieta's Method of reſolving Algebräick Equations of any Order, or Degree, by Approximation; containing an Example of the Reſolution of the Equation $x^3 - 5x^2 + 500x = 7{,}905{,}504$, *(which is reſolved by him in the 15th Problem of his Diſcourſe upon this Subject) according to his Method.*

By FRANCIS MASERES, Eſq. F. R. S.

CURSITOR BARON OF THE EXCHEQUER.

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NUMBER X.

Remarks on the Number of Negative *and* Impoſſible *Roots in Algebräick Equations.*

By WILLIAM FREND, M. A.

FELLOW OF JESUS COLLEGE, CAMBRIDGE.

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A NEW,

A NEW, EXACT AND EASIE METHOD

OF

FINDING THE ROOTS OF ANY EQUATIONS GENERALLY, AND THAT WITHOUT ANY PREVIOUS REDUCTION.

By DR. EDMUND HALLEY.

A New, Exact and Easie METHOD *of finding the* ROOTS *of any* EQUATIONS *Generally, and that without any previous Reduction. By* EDM. HALLEY. *Being Number* 210 *of the* Philosophical Transactions *published in May* 1694.

THE principal use of the *Analytick Art*, is to bring Mathematical Problems to Equations, and to exhibit those Equations in the most simple Terms that can be. But this Art would justly seem in some degree defective, and not *sufficiently Analytical*, if there were not some Methods, by the help of which the Roots (be they Lines or Numbers,) might be gotten from the Equations that are found, and so the Problems in that respect be solved. The Ancients scarce knew any thing in these Matters, beyond *Quadratick Equations*. And what they writ of the *Geometrick Construction* of solid Problems, by the help of the *Parabola*, *Cissoid*, or any other Curve, were only particular things design'd for some particular Cases. But as to *Numerical Extraction*, there is every where a profound Silence; so that whatever we perform now in this kind, is entirely owing to the Inventions of the Moderns.

And first of all, that great Discoverer and Restorer of the Modern Algebra, *Francis Vieta*, about 100 years

ſince, ſhew'd a general Method for extracting the Roots of any Equation, which he publiſh'd under the Title of, *A Numerical Reſolution of Powers*, &c. And the ſubſequent writers *Harriot*, *Oughtred*, and others, as well of our own Country as Foreigners, ought to acknowledge whatſoever they have written upon this Subject, as taken from *Vieta*. But what the Sagacity of Mr. *Newton*'s* Genius has perform'd in this buſineſs, we may rather conjecture (than be fully aſſured of) from that ſhort Specimen given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we muſt be forced to expect it, till his great Modeſty ſhall yield to the Intreaties of his Friends, and ſuffer thoſe curious Diſcoveries to ſee the Light.

Not long ſince (*viz. A. D.* 1690) that excellent Perſon M. *Joſeph Raphſon*, F. R. S. publiſh'd his *Univerſal Analyſis of Equations*, and illuſtrated his Method by plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greateſt things may be expected.

By his Example, M. *de Lagny*, an ingenious Profeſſor of Mathematicks at *Paris*, was encouraged to attempt the ſame Argument; but he, being almoſt altogether taken up in extracting the Roots of pure Powers (eſpecially the Cubick) adds but little about Affected Equations, and that pretty much perplex'd too, and not ſufficiently demonſtrated. Yet he gives two very compendious Expreſſions for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one. Ex. gr. that the ſide of the cube $aaa + b$ is between

* Afterwards Sir Iſaac Newton.

$a +$

$a + \frac{ab}{3aaa+b}$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a$.

And the root of the 5th Power $a^5 + b$, he makes $= \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ (where Note, that 'tis $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as 'tis erroneouſly printed in the French Book.) Theſe Rules were communicated to me by a Friend, I having not ſeen the Book; but, having by tryal found the goodneſs of them, and admiring the Compendium, I was willing to find out the Demonſtration: Which having done, I preſently found that the ſame Method might be accommodated to the Reſolution of all ſorts of Equations. And I was the rather inclin'd to improve theſe Rules, becauſe I ſaw that the whole thing might be explain'd in a *Synopſis*; and that by this means, at every repeated ſtep of the Calculus, the Figures already found in the Root would be at leaſt trebled; which, by all other ways *, are encreaſed but in an equal Number with the given ones. Now, the fore-mention'd Rules are eaſily demonſtrated from the Geneſis of the Cube, and the 5th Power. For, ſuppoſing the ſide of any Cube to be $= a + e$, the Cube ariſing from thence will be $aaa + 3aae + 3aee + eee$. And conſequently, if we ſuppoſe aaa to be the next leſs Cube to any given Non-cubick Number denoted by $aaa + b$, and $a + e$ to be the Cube-root of the ſaid Number, it is evident that e will be leſs than 1, and conſequently that eee will alſo be leſs than 1, or Unity, and that the remainder b, or the difference of the Numbers aaa and $aaa + b$, will be $=$ the three other Members of the Cube, to wit, $3aae +$

* This alludes to Mr. Raphſon's and Sir Iſaac Newton's Methods.

$3aee + eee$. Whence, rejecting eee upon the account of its ſmallneſs, we have $b = 3aae + 3aee$. And, ſince aae is much greater than aee, the quantity $\frac{b}{3aa}$ will not much exceed e; ſo that, putting $e = \frac{b}{3aa}$, then the quantity $\frac{b}{3aa + 3ae}$ (to which e is nearly equal) will be found $= \frac{b}{3aa + \frac{3ab}{3aa}}$, or $\frac{b}{3aa + \frac{b}{a}}$; that is, $\frac{ab}{3aaa+b}$ will be nearly $= e$. And ſo the ſide of the Cube $aaa + b$ will be nearly $= a + \frac{b}{3aa+\frac{b}{a}}$, or $a + \frac{ab}{3aaa + b}$, which is the *Rational Formula* of *M. de Legny*. But now, if aaa were the next greater Cubick Number to that given, the ſide of the Cube $aaa - b$, will, after the ſame manner, be found to be nearly $= a - \frac{b}{3aa - \frac{b}{a}}$, or $a - \frac{ab}{3aaa - b}$. And this eaſy and expeditious Approximation to the Cubick Root is (only a very ſmall matter) erroneous in point of *defect*, the quantity e, the remainder of the Root thus found, coming ſomething leſs than really 'tis *.

* I believe the expreſſions $a + \frac{b}{3aa + \frac{b}{a}}$, and $a - \frac{b}{3aa - \frac{b}{a}}$ will be found eaſier to compute to any given number of figures than the equivalent and, ſeemingly, more ſimple expreſſions $a + \frac{ab}{3aaa+b}$ and $a - \frac{ab}{3aaa-b}$, and more eſpecially when a conſiſts of four, or more, decimal figures. F. M.

As for the *Irrational Formula*, 'tis deriv'd from the ſame Principle, *viz.* $b = 3aae + 3aee$, or $\frac{b}{3a} = ae + ee$, and ſo $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} = \frac{1}{2}a + e$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a = a + e$, the Root ſought. Alſo the ſide of the Cube $aaa - b$, after the ſame manner, will be found to be $\frac{1}{2}a + \sqrt{\frac{1}{4}aa - \frac{b}{3a}}$. And this *Formula* comes ſomething nearer to the Scope, being erroneous in point of *exceſs*, as the other was in *defect*, and is more accommodated to the ends of Practice, ſince the Reſtitution of the Calculus, is nothing elſe but the continual addition or ſubtraction of the Quantity $\frac{eee}{3a}$ according as the quantity e can be known. So that we ſhould rather write $\sqrt{\frac{1}{4}aa + \frac{b - eee}{3a}} + \frac{1}{2}a$, in the former caſe, and, in the latter, $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{eee - b}{3a}}$. But by either of the two *Formula*'s, the Figures already known in the Root to be extracted, are at leaſt tripled; which I conclude will be very grateful to all the Students in Arithmetick; and I congratulate the Inventor upon the account of his Diſcovery.

Examples of the Uſe of theſe Rules in the Extraction of the Cube-Roots of Numbers.

But that the uſe of theſe Rules may be the better perceiv'd, I think it proper to ſubjoin an Example or two.

Let it be propos'd to find the ſide of the double Cube, or let $aaa + b$ be $= 2$. Here a is $= 1$, and conſequently a^3 is $= 1$, and b is $= 1$, and $\frac{a}{2}$ is $(= \frac{1}{2}) = 0.50$, and $\frac{b}{3a}$ is $= \frac{1}{3}$, and $\frac{1}{4}aa + \frac{b}{3a}$ is $(= \frac{1}{4} \times 1 + \frac{1}{3} = \frac{1}{4} + \frac{1}{3}) = \frac{7}{12}$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}}$ is $(= \sqrt{\frac{7}{12}} = \sqrt{0.5833}) = 0.76$, and $\frac{a}{2} + \sqrt{\frac{1}{4}aa + \frac{b}{3a}}$ is $(= 0.50 + 0.76) = 1.26$, which will be found to be the true ſide nearly. Now, the Cube of 1.26, is 2.000,376, and ſo $0.63 + \sqrt{.3969 - \frac{.000,376}{3.78}}$, or $0.63 + \sqrt{.396800,529100,5291}$ $= 1.259,921,049,895 -$; which in 13 Figures, gives the ſide of the double Cube, with very little trouble, *viz.* By only one diviſion, and the extraction of the ſquare Root; when-as by the common way of working, how much pains it would have coſt, the Skilful very well know. This Calculus a Man may continue as far as he pleaſes, by encreaſing the Square by the addition of the quantity $\frac{eee}{3a}$; which Correction, in this caſe, will give only the encreaſe of Unity in the 14th Figure of the Root.

Examp. 2*d.* Let it be propos'd to find the ſide of a Cube equal to that Engliſh Meaſure commonly call'd a Gallon, which contains 231 ſolid Ounces *. The next leſs Cube is 216, whoſe ſide $6 = a$, and the remainder is $15 = b$; and ſo, for the firſt Approximation, we have

* Or Cubick Inches.

$\frac{a}{3} + \sqrt{\frac{aa}{4} + \frac{b}{3a}}$ $(= \frac{6}{2} + \sqrt{\frac{36}{4} + \frac{15}{3\times 6}} = 3 + \sqrt{9 + \frac{5}{6}} = 3 + \sqrt{9 + 0.8333}) = 3 + \sqrt{9.8333}$ = the Root. And since $\sqrt{9.8333}\ldots$ is $3.1358\ldots$, 'tis plain that $6.358 = a + e$. Now, let $6.1358 = a$; and we shall then have for its Cube - - 231.000,853,894,712, and, according to the Rule, $3.0679 + \sqrt{9.412,010,41 - \frac{0.000,853,894,712}{18.4074}}$ is most accurately equal to the side of the given Cube, which within the space of an Hour, I determin'd by Calculation to be 6.135,792,439,661,958,97, which is exact in the 17th Figure, and defective in the 18th. And this *Formula* is deservedly preferable to the *Rational Formula*, upon the account of the great Divisor $3aaa + b$, or $3aaa - b$, in the Rational Formula, which is not to be managed without a great deal of Labour; whereas the extraction of the square-root proceeds much more easily, as manifold Experience has taught me.

Of the Extraction of the Fifth Root of a Given Number.

But the Rule for the Root of a pure Surfolid, or the 5th Power, is of something a higher Enquiry, and does much more perfectly yet, do the business: for it does at least Quintuple the given Figures in the Root; neither is the Calculus very large or operose. Tho' the Author no-where shews his method of Invention, or any Demonstration, altho' it seems to be very much wanting; especially since all things are not right in the printed Book, which may easily deceive the Unskilful. Now the 5th power of the side $a + e$ is compos'd of these Members, $a^5 + 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4 + e^5 = a^5 + b$;

from whence b is $= 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4$, rejecting e^5 becauſe of its ſmallneſs. Whence $\frac{b}{5a} = a^3e + 2a^2e^2 + 2ae^3 + e^4$, and (adding on both ſides $\frac{1}{4}a^4$, and then extracting the ſquare-roots of both ſides,) we ſhall have $\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} = \sqrt{\frac{1}{4}a^4 + a^3e + 2a^2e^2 + 2ae^3 + e^4}$ $= \frac{1}{2}aa + ae + ee$. Then (ſubtracting $\frac{1}{2}a$ from both ſides, and extracting the ſquare-roots of the remainders,) $\frac{1}{2}a + e$ will be $= \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$; to which if $\frac{1}{2}a$ be added, then will $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ = the fifth root of the Number $a^5 + b$. But, if it had $a^5 - b$ (the quantity a being too great) the Rule would have been thus, $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 - \frac{b}{5a}} - \frac{1}{4}aa}$. And this Rule approaches wonderfully, ſo that there is hardly any need of Reſtitution.

Of the Extraction of all other Roots of Given Numbers by Means of ſimilar Rules to thoſe above given for the Extraction of the Third and Fifth Roots.

But, while I conſidered theſe things with my ſelf, I lit upon a General Method for the *Formula*'s of all Powers whatſoever, and (which being handſome and conciſe enough) I thought I would not conceal it from the Publick.

Theſe Formula's, (as well the *Rational*, as the *Irrational* ones) are thus.

$\sqrt{aa + b}$

$\sqrt{aa + b} = \sqrt{aa + b}$, or $a + \frac{ab}{2aa + \frac{1}{2}b}$.

$\sqrt[3]{a^3 + b} = \frac{1}{3}a + \sqrt{\frac{1}{9}aa + \frac{b}{3a}}$, or $a + \frac{ab}{3aaa + b}$.

$\sqrt[4]{a^4 + b} = \frac{2}{4}a + \sqrt{\frac{1}{16}aa + \frac{b}{6aa}}$, or $a + \frac{ab}{4a^3 + \frac{3}{2}b}$.

$\sqrt[5]{a^5 + b} = \frac{3}{5}a + \sqrt{\frac{1}{25}aa + \frac{b}{10a^3}}$, or $a + \frac{ab}{5a^4 + 2b}$.

$\sqrt[6]{a^6 + b} = \frac{4}{6}a + \sqrt{\frac{1}{36}aa + \frac{b}{15a^4}}$, or $a + \frac{ab}{6a^5 + \frac{5}{2}b}$.

$\sqrt[7]{a^7 + b} = \frac{5}{7}a + \sqrt{\frac{1}{49}aa + \frac{b}{21a^5}}$, or $a + \frac{ab}{7a^6 + 3b}$ *.

And ſo alſo of the other higher Powers. But if a were aſſumed bigger, than the Root ſought (which is done

* The general expreſſions for theſe values of the root of any given number are as follows. Let the given number be called N, and (m being any whole number whatſoever,) let a near value of it's mth root that is leſs than the truth be called a. Then will the mth root of the given number N be very nearly equal either to the rational expreſſion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or to the irrational expreſſion $a + \sqrt{\left(\frac{aa}{\overline{m-1}^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}\right)} - \frac{a}{m-1}$; as is ſhewn at large in my Explanation of Monſieur de Lagny's method of extracting the Roots of Numbers, publiſhed with other Mathematical Tracts in a large octavo volume, in the year 1795, pages 507, 508, &c, 516. F. M.

with

with ſome advantage, as often as the Power to be reſolved, is much nearer to the Power of the *next greater* whole Number, than to that of the *next leſs*) in this caſe, *Mutatis Mutandis*, we ſhall have the ſame Expreſſions of the Roots, *viz.*

$\sqrt{aa-b} = \sqrt{aa-b}$, or $a - \frac{ab}{2aa-\frac{1}{2}b}$.

$\sqrt[3]{a^3-b} = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - \frac{b}{3a}}$, or $a - \frac{ab}{3a^3-b}$.

$\sqrt[4]{a^4-b} = \frac{2}{3}a + \sqrt{\frac{1}{9}aa - \frac{b}{6aa}}$, or $a - \frac{ab}{4a^4-\frac{1}{2}b}$.

$\sqrt[5]{a^5-b} = \frac{3}{4}a + \sqrt{\frac{1}{16}aa - \frac{b}{10a^3}}$, or $a - \frac{ab}{5a^5-2b}$.

$\sqrt[6]{a^6-b} = \frac{4}{5}a + \sqrt{\frac{1}{25}aa - \frac{b}{15a^4}}$, or $a - \frac{ab}{6a^6-\frac{5}{2}b}$.

$\sqrt[7]{a^7-b} = \frac{5}{6}a + \sqrt{\frac{1}{36}aa - \frac{b}{21a^5}}$, or $a - \frac{ab}{7a^7-3b}$ *.

* The general expreſſions of theſe values of the root of any given number are as follows. Let the given number be called N, and (m being any whole number whatſoever,) let a near value of its mth root that is greater than the truth be called a. Then will the mth root of the given number N be very nearly equal either to the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or to the irrational expreſſion $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$; as is ſhewn in the ſaid large octavo volume of Tracts, in pages 536, 537, - - - 546, where the ſubject is very fully treated, and illuſtrated by examples. F. M.

And

And within theſe two Terms the true Root is ever found, being ſomething nearer to the *Irrational* than the *Rational* Expreſſion. But the quantity *e* found by the *Irrational Formula* is always too great, as the quotient reſulting from the Rational Formula is always too little. And conſequently, if we have + *b*, the *Irrational Formula* gives the Root ſomething greater than it ſhould be, and the *Rational* ſomething leſs. But contrary-wiſe, if it be — *b*.

And thus much may ſuffice to be ſaid concerning the extraction of the Roots of pure Powers; which notwithſtanding, for common Uſes, may be had much more eaſily by the help of the Logarithms. But when a Root is to be determin'd very accurately, and the Logarithmick Tables will not reach ſo far, then we muſt neceſſarily have recourſe to theſe, or ſuch like Methods. Farther; the Invention and Contemplation of theſe Formulæ, leading me to a certain Univerſal Rule for Adfected Equations, (which I hope will be of uſe to all the Students in *Algebra* and *Geometry*,) I was willing here to give ſome account of this Diſcovery, which I will do with all the perſpicuity I can. I had given, in No. 188 of the *Tranſactions*, a very eaſy and general conſtruction of all Adfected Equations, not exceeding the Biquadratick Power; from which time I had a very great deſire of doing the ſame in Numbers. But quickly after, Mr. *Raphſon* ſeem'd in great meaſure to have ſatisfy'd this Deſire, till Mr. *de Lagny*, by what he had perform'd in his Book, intimated that the thing might be done more compendiouſly yet. Now, my Method is thus.

Of

Of the Reſolution of Affected Equations.

Let z, the root of any Equation, be imagin'd to be compos'd of the parts $a + $ or $- e$, of which let a be aſſum'd as near z as is poſſible; which is notwithſtanding not *neceſſary*, but only *commodious*. Then from the Quantity $a + e$ or $a - e$, let there be form'd all the Powers of z, found in the Equation, and let the Numerical Co-efficients of the ſaid Powers in the Terms of the Equation be reſpectively affixed to them: Then let the Power to be *reſolved*, (in the new, or transformed, Equation ariſing from this ſubſtitution of $a + e$, or $a - e$, inſtead of z, in the Terms of the original Equation,) be ſubtracted from the ſum of the known Parts (in the firſt Column of Terms in the ſaid new Equation, amongſt which the unknown Quantity e is not found) which they call the *Homogeneum Comparationis*, and let the difference be $\pm b$. In the next place, take the ſum of all the Co efficients of e in the ſecond Column, to which put $= s$. Laſtly, in the third Column let there be put down the ſum of all the Co-efficients of ee, which ſum call t. Then will the Root z ſtand thus in the *Rational Formula*, *viz.* $z = a + \frac{sb}{ss \pm tb}$; and thus in the *Irrational Formula*, *viz.* $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illuſtrate by ſome Examples. And inſtead of an *Inſtrument*, let this *Table* ſerve, which ſhews the Geneſis of the ſeveral Powers of $a \pm e$, and if need be, may eaſily be continued farther; which for its uſe I may rightly call *A General Analytical Speculum*. The forementioned Powers ariſing from a continual Multiplication by $a + e$ $(= z)$ come out thus with their adjoyned Co-efficients.

TABELLA

TABELLA POTESTATUM,
or
A GENERAL ANALYTICAL SPECULUM.

	s	t	u	w	x	y
$lz^7 = la^7$	$+ 7la^6e$	$+ 21la^5ee$	$+ 35la^4e^3$	$+ 35la^3e^4$	$+ 21la^2e^5$	$+ 7lae^6 + le^7$
$kz^6 = ka^6$	$+ 6ka^5e$	$+ 15ka^4ee$	$+ 20ka^3e^3$	$+ 15ka^2e^4$	$+ 6kae^5$	$+ ke^6$
$bz^5 = ba^5$	$+ 5ba^4e$	$+ 10ba^3ee$	$+ 10ba^2e^3$	$+ 5bae^4$	$+ be^5$	
$gz^4 = ga^4$	$+ 4ga^3e$	$+ 6ga^2ee$	$+ 4gae^3$	$- ge^4$		
$fz^3 = fa^3$	$+ 3fa^2e$	$+ 3faee$	$+ fe^3$			
$dz^2 = da^2$	$+ 2dae$	$+ dee$				
$cz = ca$	$+ ce$					

But now, if it be $a - e = z$, the Table is compos'd of the ſame Members; only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, are Affirmative. Alſo let the ſum of the Co-efficients of the ſide e be $= s$; the ſum of the Co-efficients of the Square ee be $= t$, the ſum of the Co-efficients of e^3 be $= u$; of $e^4 = w$; of $e^5 = x$; of $e^6 = y$, &c. But now, ſince e is ſuppos'd to be only a ſmall part of the Root that is to be enquir'd, all the Powers of e will be much leſs than the correſpondent Powers of a, and ſo, for the firſt Hypotheſis, all the ſuperiour ones may be rejected; and forming a new Equation, by ſubſtituting $a \pm e = z$, we ſhall have (as was ſaid) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

Example I.

Let the Equation $z^4 - 3z^2 + 75z = 10{,}000$, be propos'd. For the firſt Hypotheſis, let $a = 10$, and ſo we have this Equation,

$z^4 =$

$$
\begin{array}{rrrrrr}
z^4 = & +a^4 & 4a^3e & +6a^2e^2 & 4ae^3 & +e^4 \\
-dz^2 = & -da^2 & 2dae & -de^2 & & \\
+cz = & +ca & ce & & & \\
= & +10000 & 4000e & +600ee & 40e^3 & +e^4 \\
& -300 & 60e & -3ee & & \\
& +750 & 75e & & & \\
& =10000 & & & &
\end{array}
$$

$$
\begin{array}{rrrrrr}
 & +450 & 4015e & +597ee & 40e^3 & +e^4 = 0 \\
 & & s & t & u &
\end{array}
$$

The Signs + and − with reſpect to the Quantities e and e^3, are left as doubtful, till it be known whether e be Negative or Affirmative; which thing creates ſome difficulty, ſince it ſometimes happens, (to wit, in Equations that have ſeveral Roots,) that the *Homogeneum Comparationis* (as they term it) is encreaſed by the diminution of the quantity a, and on the contrary, that, when a is encreaſed, the ſaid *Homogeneum* is diminiſhed. But the Sign of e is determin'd from the Sign of the Quantity b. For, taking away the *Reſolvend* from the *Homogeneum* formed of a, the Sign of se (and conſequently of the prevailing Parts in the compoſition of it) will always be contrary to the Sign of the difference b. Whence 'twill be plain, whether it muſt be $+e$, or $-e$; and conſequently whether a be taken greater or leſs than the *True Root*. Now the quantity e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, when b and t have the ſame Sign; but, when the Signs are different, e is $= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$. But after it is found that it will be

be $-e$, let the Powers e, e^3, and e^5, &c. in the Affirmative Members of the Equation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Signs. On the other hand, if it be $+e$, let thoſe forementionʼd Powers be made Affirmative in the Affirmative, and Negative in the Negative Members of the Equation.

Now we have in this Example of ours, 10,450 inſtead of the Reſolvend 10,000, or $b = +450$; whence it is plain that a is taken greater than the Truth, and conſequently, that 'tis $-e$. Hence the Equation comes to be, $10{,}450 - 4015e + 597ee - 40e^3 + e^4 = 10{,}000$. That is, $450 - 4015e + 597ee = 0$; and ſo $450 = 4015e - 597ee$, or $b = se - tee$; whoſe Root e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, or $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, in the preſent caſe, $e = \frac{2007\frac{1}{2} - \sqrt{3{,}761{,}406\frac{1}{4}}}{597}$, from whence we have the Root ſought, 9.886, which is near the Truth. But then, ſubſtituting this for a ſecond Suppoſition, there comes $a + e = z$, moſt accurately 9.886,260,393,649,5 .... ſcarce exceeding the Truth by 2 in the laſt Figure, *viz.* when $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= e$. And this (if need be) may be yet much farther verified, by ſubtracting (if it be $+e$) the quantity $\frac{\frac{1}{2}ue^3 + \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss + tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ to that

that Root. Which Compendium is ſo much the more valuable, in that ſometimes from the firſt Suppoſition alone, but always from the ſecond, a Man may continue the Calculus (keeping the ſame Co-efficients) as far as he pleaſes. It may be noted that the fore-mentioned Equation has alſo a Negative Root, *viz.* $z = 10.26\ldots$ which any one, that has a mind, may determine more accurately.

Example II.

Suppoſe $z^3 - 17z^2 + 54z = 350$, and let $a = 16$. Then according to the preſcript of the Rule,

$$\begin{aligned} + z^3 &= a^3 + 3a^2e + 3ae^2 + e^3 \\ - dz^2 &= - da^2 - 2dae - de^2 \\ + cz &= ca + ce \end{aligned}$$

$$\begin{array}{rccccc} & & b & & s & t \\ \text{That is,} & + & 1000 & + 300e + & 30e^2 & + e^3 \\ & - & 1700 & - 340e - & 17e^2 & \\ & + & 540 & + 54e & & \\ & = & 350 & & & \end{array}$$

Or, $-510 + 14e + 13ee + e^3 = 0$. Now, ſince we have -510, it is plain, that a is aſſumed leſs than the Truth, and conſequently that e is Affirmative. And from the Equation $510 = 14e + 13e^2$ comes $e = \frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \frac{\sqrt{6679} - 7}{13} \left(= \frac{81.7 - 7}{13} = \frac{74.7}{13}\right)$ $= 5.7$. Whence z (or $10 + e$,) is $(= 10 + 5.7) = 15.7\ldots$, which is too much, becauſe of a taken wide. Therefore, Secondly, let $a = 15$, and by the like way of Reaſoning, we

we ſhall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$,

and conſequently $z = 14.954{,}068$. If the Operation were to be repeated the third time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ inſtead of tb, or ſubtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to the Root before found, will preſently obtain his end.

Note, the Equation propoſed is not explicable by any other Root, becauſe the *Reſolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

Example III.

Let us take the Equation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uſes, in Cap. 62 of his *Algebra*, in the Reſolution of a very difficult Arithmetical Problem, where by *Vieta*'s Method he has obtain'd the root moſt accurately; and Mr. *Raphſon* brings it alſo as an Example of his Method, in Pages 25, 26. Now this Equation is of the form which may have ſeveral Affirmative Roots, and (which increaſes the difficulty of reſolving it) the *Co-efficients* are very great in reſpect of the *Reſolvend* given.

But, that it may be the eaſier managed, let it be divided, and, according to the known Rules of *Pointing*,

let $-z^4 + 8z^3 - 20z^2 + 15z = 0{,}5$ (where the quantity z is $\frac{1}{10}$ of z in the Equation propoſed) and for the firſt Suppoſition, let $a = 1$. Then will $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0{,}5$ be $= 0$; that is, $1\frac{1}{2}$ will be $= 5e + 2ee$. Here therefore b is $= 1\frac{1}{2}$ or 1.5, and s is $= 5$, and t is $= 2$; and conſequently $\frac{1}{4}ss$ is $= \frac{25}{4}$, and bt is $(= 1.5 \times 2 = 3.0) = \frac{12}{4}$, and $\frac{1}{4}ss + bt$ is $= \frac{25}{4} + \frac{12}{4} = \frac{37}{4}$, and $\sqrt{\frac{1}{4}ss + bt}$ is $= \frac{\sqrt{37}}{2}$, and $\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}$ is $= \frac{\sqrt{37}}{2} - \frac{5}{2} = \frac{\sqrt{37} - 5}{2}$, and $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}}{t}$ is $= \frac{\frac{\sqrt{37} - 5}{2}}{2} = \frac{\sqrt{37} - 5}{4}$; and hence e $(= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t})$ is $= \frac{\sqrt{37} - 5}{4}$ $(= \frac{6.08 - 5}{4} = \frac{1.08}{4})$ $= 0.27$, and ſo $a + e$, or z, is $= 1 + 0.27$, or 1.27; whence 'tis manifeſt that $(10 \times 1.27$, or) 12.7 is near the true Root of the Equation propoſed. Now, Secondly, let us ſuppoſe $z = 12.7$, and then, according to the directions of the Table of Powers, there ariſes

b	s	t	u
$-$ 26,014.4641	$-$ 8,193.532e	$-$ 967.74e^2	$-$ 50.8e^3 $-e^4$
$+$ 163,870.640	$+$ 38,709.60e	$+$ 3048 e^2	$+$ 80 e^3
$-$ 322,257.42	$-$ 50,749.2 e	$-$ 1998 e^2	
$+$ 189,699.9	$+$ 14,937. e		
$-$ 5000.			

That

That is, $+ 298.6559 - 5296.132e + 82.26e^2 + 29.2e^3 - e^4 = 0$; And ſo $- 298.6559 = - 5296.132e + 82.26ee$, whoſe Root e (being, according to the Rule, $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$) comes to $\frac{2648.066 - \sqrt{6,987,686.106,022}}{82.26} = .056,440,803,31\ldots.$ $= e$; which is leſs than the Truth. But, that it may be corrected, 'tis to be conſider'd that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{.002,620,1\ldots..}{2643.423\ldots..}$ is .000,000,99, and conſequently e corrected is $= 0.056,441,794,48$. And, if you deſire yet more Figures of the Root, from the e corrected let there be made $tue^3 - te^4 = 0.431,056,024,23\ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or (which is all one,) $\frac{2648.066 - \sqrt{6,987,685.674,965,975,77\ldots.}}{82,26}$, $=$ $.056,441,794,480,744,02 = e$; whence $a + e$, $= z$ the Root, is moſt accurately 12.756,441,794,480,744,02... as Dr. *Wallis* found in the fore-mentioned Place; where it may be obſerv'd, that the repetition of the *Calculus* does ever triple the true Figures in the aſſumed a, which the firſt correction, or $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$ does quintuple; which is alſo commodiouſly done by the *Logarithms*. But the other Correction, after the firſt, does alſo double the number of Figures, ſo that it renders the *aſſumed* altogether Seven-fold; yet the firſt Correction is abundantly ſufficient for Arithmetical uſes, for the moſt part.

But, as to what is ſaid concerning the number of Figures rightly taken in the Root, I would have it underſtood ſo that, when a is but $\frac{1}{10}$ part diſtant from the true Root, then the firſt Figure is rightly aſſumed; if it be within $\frac{1}{100}$ part, then the two firſt Figures are rightly aſſumed; and, if within $\frac{1}{1000}$ part, then the three firſt are ſo; which conſequently, managed according to our Rule, do preſently become nine Figures.

It remains now that I add ſomething concerning our *Rational Formula*, viz. $e = \frac{sb}{ss \pm tb}$, which ſeems expeditious enough, and is not much inferiour to the former, ſince it will triple the given Number of Figures. Now, when we ſhall have formed an Equation from $a \pm e = z$, as before, it will preſently appear, whether a be taken greater or leſs than the Truth; ſince se ought always to have a Sign contrary to the Sign of the difference of the *Reſolvend* and its *Homogeneum Comparationis*, produced from a. Then, ſuppoſing $+ b + se + a - tee = 0$, the Diviſor is $ss - tb$, as often as t and b have the ſame Signs; but it is $ss + bt$, when they have different ones. But it ſeems moſt commodious for Practice to write the Theorem thus, $e = \frac{b}{s \pm \frac{tb}{s}}$; ſince in this way the thing is done by one Multiplication and two Diviſions, which otherwiſe would require three Multiplications and one Diviſion.

Let us take now one Example of this Method, from the Root (of the fore-mention'd Equation) 12.7 . . . . , where

298.6559

$$\begin{array}{llll} 298.6559 & - 5296.132\,e & + 82.26\,ee & + 29.2\,e^3 \\ + b & - s & + t & + u \end{array}$$

$- e^4$ is $= o$, and ſo $\frac{b}{s - \frac{tb}{s}}$ is $= e$; that is, let us make the following Proportion, to wit, as s is to t, ſo is b to $\frac{tb}{s}$, or as 5296.132 is to 82.26, ſo is 298.6559 to $\frac{tb}{s}$; which will therefore be $(= \frac{82.26 \times 298,6559}{5296.132})$ $= 4.638,75$; and conſequently the Diviſor $s - \frac{tb}{s}$ will be $(= 5296.132 - 4.638,75) = 5291.493,25$, and $\frac{b}{s - \frac{tb}{s}}$ will be $(= \frac{298.6559}{5291.493,25}) = 0.056,441$; that is, e will be $= 0.056,441$, in which all the Figures are true. And conſequently z, or $a + e$, or the Root of the propoſed Equation $z^4 - 80z^3 + 1998z^2 - 14,937z + 5000 = o$, will be $(= 12.7 + 0.056,441) = 12.756,441$; of which all the eight Figures are true. But this *Rational Formula* $e = \frac{sb}{ss \pm tb}$, or $\frac{b}{s \pm \frac{tb}{s}}$, cannot be corrected, as the foregoing *Irrational* one was; and ſo, if more Figures of the Root are deſired, 'tis the beſt to make a new Suppoſition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly ſatisfie even the moſt ſcrupulous.

Of the Reſolution of Affected Equations.

Let z, the root of any Equation, be imagin'd to be compos'd of the parts $a +$ or $- e$, of which let a be aſſum'd as near z as is poſſible; which is notwithſtanding not *neceſſary*, but only *commodious*. Then from the Quantity $a + e$ or $a - e$, let there be form'd all the Powers of z, found in the Equation, and let the Numerical Co-efficients of the ſaid Powers in the Terms of the Equation be reſpectively affixed to them: Then let the Power to be *reſolved*, (in the new, or transformed, Equation ariſing from this ſubſtitution of $a + e$, or $a - e$, inſtead of z, in the Terms of the original Equation,) be ſubtracted from the ſum of the known Parts (in the firſt Column of Terms in the ſaid new Equation, amongſt which the unknown Quantity e is not found) which they call the *Homogeneum Comparationis*, and let the difference be $\pm b$. In the next place, take the ſum of all the Co efficients of e in the ſecond Column, to which put $= s$. Laſtly, in the third Column let there be put down the ſum of all the Co-efficients of ee, which ſum call t. Then will the Root z ſtand thus in the *Rational Formula*, *viz.* $z = a + \frac{sb}{ss \mp tb}$; and thus in the *Irrational Formula*, *viz.* $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illuſtrate by ſome Examples. And inſtead of an *Inſtrument*, let this *Table* ſerve, which ſhews the Geneſis of the ſeveral Powers of $a \pm e$, and if need be, may eaſily be continued farther; which for its uſe I may rightly call *A General Analytical Speculum*. The forementioned Powers ariſing from a continual Multiplication by $a + e$ $(= z)$ come out thus with their adjoyned Co-efficients.

TABELLA

TABELLA POTESTATUM,
or
A GENERAL ANALYTICAL SPECULUM.

$$\begin{array}{lllllll} & s & t & u & w & x & y \\ lz^7 = la^7 & +7la^6e & +21la^5ee & +35la^4e^3 & +35la^3e^4 & +21la^2e^5 & +7lae^6+le^7 \\ kz^6 = ka^6 & +6ka^5e & +15ka^4ee & +20ka^3e^3 & +15ka^2e^4 & +6kae^5 & +ke^6 \\ bz^5 = ba^5 & +5ba^4e & +10ba^3ee & +10ba^2e^3 & +5bae^4 & +be^5 & \\ gz^4 = ga^4 & +4ga^3e & +6ga^2ee & +4gae^3 & -ge^4 & & \\ fz^3 = fa^3 & +3fa^2e & +3faee & +fe^3 & & & \\ dz^2 = da^2 & +2dae & +dee & & & & \\ cz = ca & +ce & & & & & \end{array}$$

But now, if it be $a - e = z$, the Table is compos'd of the ſame Members; only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, are Affirmative. Alſo let the ſum of the Co-efficients of the ſide e be $= s$; the ſum of the Co-efficients of the Square ee be $= t$, the ſum of the Co-efficients of e^3 be $= u$; of $e^4 = w$; of $e^5 = x$; of $e^6 = y$, &c. But now, ſince e is ſuppos'd to be only a ſmall part of the Root that is to be enquir'd, all the Powers of e will be much leſs than the correſpondent Powers of a, and ſo, for the firſt Hypotheſis, all the ſuperiour ones may be rejected; and forming a new Equation, by ſubſtituting $a \pm e = z$, we ſhall have (as was ſaid) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

Example I.

Let the Equation $z^4 - 3z^2 + 75z = 10{,}000$, be propos'd. For the firſt Hypotheſis, let $a = 10$, and ſo we have this Equation,

$z^4 =$

$$
\begin{array}{rrrrrr}
z^4 = & +a^4 & 4a^3e & +6a^2e^2 & 4ae^3 & +e^4 \\
-dz^2 = & -da^2 & 2dae & -de^2 & & \\
+cz = & +ca & ce & & & \\
= & +10000 & 4000e & +600ee & 40e^3 & +e^4 \\
& -300 & 60e & -3ee & & \\
& +750 & 75e & & & \\
& =10000 & & & &
\end{array}
$$

$$
\begin{array}{rrrrr}
+450 & 4015e & +597ee & 40e^3 & +e^4 = 0 \\
 & s & t & u &
\end{array}
$$

The Signs + and − with reſpect to the Quantities e and e^3, are left as doubtful, till it be known whether e be Negative or Affirmative; which thing creates ſome difficulty, ſince it ſometimes happens, (to wit, in Equations that have ſeveral Roots,) that the *Homogeneum Comparationis* (as they term it) is encreaſed by the diminution of the quantity a, and on the contrary, that, when a is encreaſed, the ſaid *Homogeneum* is diminiſhed. But the Sign of e is determin'd from the Sign of the Quantity b. For, taking away the *Reſolvend* from the *Homogeneum* formed of a, the Sign of se (and conſequently of the prevailing Parts in the compoſition of it) will always be contrary to the Sign of the difference b. Whence 'twill be plain, whether it muſt be $+e$, or $-e$; and conſequently whether a be taken greater or leſs than the *True Root*. Now the quantity e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, when b and t have the ſame Sign; but, when the Signs are different, e is $= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$. But after it is found that it will be

be $-e$, let the Powers e, e^3, and e^5, &c. in the Affirmative Members of the Equation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Signs. On the other hand, if it be $+e$, let thoſe foremention'd Powers be made Affirmative in the Affirmative, and Negative in the Negative Members of the Equation.

Now we have in this Example of ours, 10,450 inſtead of the Reſolvend 10,000, or $b = +450$; whence it is plain that a is taken greater than the Truth, and conſequently, that 'tis $-e$. Hence the Equation comes to be, $10{,}450 - 4015e + 597ee - 40e^3 + e^4 = 10{,}000$. That is, $450 - 4015e + 597ee = 0$; and ſo $450 = 4015e - 597ee$, or $b = se - tee$; whoſe Root e is $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, or $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, in the preſent caſe, $e = \frac{2007\frac{1}{2} - \sqrt{3{,}761{,}406\frac{1}{4}}}{597}$, from whence we have the Root ſought, 9.886, which is near the Truth. But then, ſubſtituting this for a ſecond Suppoſition, there comes $a + e = z$, moſt accurately 9.886,260,393,649,5 .... ſcarce exceeding the Truth by 2 in the laſt Figure, *viz.* when $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= e$. And this (if need be) may be yet much farther verified, by ſubtracting (if it be $+e$) the quantity $\frac{\frac{1}{2}ue^3 + \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss + tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ to that

that Root. Which Compendium is ſo much the more valuable, in that ſometimes from the firſt Suppoſition alone, but always from the ſecond, a Man may continue the Calculus (keeping the ſame Co-efficients) as far as he pleaſes. It may be noted that the fore-mentioned Equation has alſo a Negative Root, *viz.* $z = 10.26\ldots$, which any one, that has a mind, may determine more accurately.

Example II.

Suppoſe $z^3 - 17z^2 + 54z = 350$, and let $a = 16$. Then according to the preſcript of the Rule,

$$\begin{aligned} +\ z^3 &= a^3 + 3a^2e + 3ae^2 + e^3 \\ -\ dz^2 &= -\ da^2 - 2dae - de^2 \\ +\ cz &= ca + ce \end{aligned}$$

That is,

	b	s	t	
+	1000	+ $300e$	+ $30e^2$	+ e^3
−	1700	− $340e$	− $17e^2$	
+	540	+ $54e$		
=	350			

Or, $-510 + 14e + 13ee + e^3 = 0$. Now, ſince we have -510, it is plain, that a is aſſumed leſs than the Truth, and conſequently that e is Affirmative. And from the Equation $510 = 14e + 13e^2$ comes $e = \frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \frac{\sqrt{6679} - 7}{13}$ $(= \frac{81.7 - 7}{13} = \frac{74.7}{13})$ $= 5.7$. Whence z (or $10 + e$,) is $(= 10 + 5.7) = 15.7\ldots$, which is too much, becauſe of a taken wide. Therefore, Secondly, let $a = 15$, and by the like way of Reaſoning, we

we ſhall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$,

and conſequently $z = 14.954,068$. If the Operation were to be repeated the third time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ inſtead of tb, or ſubtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to the Root before found, will preſently obtain his end.

Note, the Equation propoſed is not explicable by any other Root, becauſe the *Reſolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

Example III.

Let us take the Equation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uſes, in Cap. 62 of his *Algebra*, in the Reſolution of a very difficult Arithmetical Problem, where by *Vieta*'s Method he has obtain'd the root moſt accurately; and Mr. *Raphſon* brings it alſo as an Example of his Method, in Pages 25, 26. Now this Equation is of the form which may have ſeveral Affirmative Roots, and (which increaſes the difficulty of reſolving it) the *Co-efficients* are very great in reſpect of the *Reſolvend* given.

But, that it may be the eaſier managed, let it be divided, and, according to the known Rules of *Pointing*,

let $-z^4 + 8z^3 - 20z^2 + 15z = 0{,}5$ (where the quantity z is $\frac{1}{10}$ of z in the Equation propoſed) and for the firſt Suppoſition, let $a = 1$. Then will $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0{,}5$ be $= 0$; that is, $1\frac{1}{2}$ will be $= 5e + 2ee$. Here therefore b is $= 1\frac{1}{2}$ or 1.5, and s is $= 5$, and t is $= 2$; and conſequently $\frac{1}{4}ss$ is $= \frac{25}{4}$, and bt is $(= 1.5 \times 2 = 3.0) = \frac{12}{4}$, and $\frac{1}{4}ss + bt$ is $= \frac{25}{4} + \frac{12}{4} = \frac{37}{4}$, and $\sqrt{\frac{1}{4}ss + bt}$ is $= \frac{\sqrt{37}}{2}$, and $\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}$ is $= \frac{\sqrt{37}}{2} - \frac{5}{2} = \frac{\sqrt{37} - 5}{2}$, and $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{s}{2}}{t}$ is $= \frac{\frac{\sqrt{37}-5}{2}}{2} = \frac{\sqrt{37} - 5}{4}$; and hence e $(= \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t})$ is $= \frac{\sqrt{37} - 5}{4}$ $(= \frac{6.08 - 5}{4} = \frac{1.08}{4})$ $= 0.27$, and ſo $a + e$, or z, is $= 1 + 0.27$, or 1.27; whence 'tis manifeſt that $(10 \times 1.27$, or) 12.7 is near the true Root of the Equation propoſed. Now, Secondly, let us ſuppoſe $z = 12.7$, and then, according to the directions of the Table of Powers, there ariſes

b	s	t	u
$-$ 26,014.4641	$-$ 8,193.532e	$-$ 967.74e^2	$-$ 50.8e^3 $-e^4$
$+$ 163,870.640	$+$ 38,709.60e	$+$ 3048 e^2	$+$ 80 e^3
$-$ 322,257.42	$-$ 50,749.2 e	$-$ 1998 e^2	
$+$ 189,699.9	$+$ 14,937. e		
$-$ 5000.			

That

That is, $+ 298.6559 - 5296.132e + 82.26e^2 + 29.2e^3 - e^4 = 0$; And ſo $- 298.6559 = - 5296.132e + 82.26ee$, whoſe Root e (being, according to the Rule, $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$) comes to $\frac{2648.066 - \sqrt{6,987,686.106,022}}{82.26} = .056,440,803,31\ldots.$ $= e$; which is leſs than the Truth. But, that it may be corrected, 'tis to be conſider'd that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{.002,620,1\ldots.}{2643.423\ldots.}$ is .000,000,99, and conſequently e corrected is $= 0.056,441,794,48$. And, if you deſire yet more Figures of the Root, from the e corrected let there be made $tue^3 - te^4 = 0.431,056,024,23\ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or (which is all one,) $\frac{2648.066 - \sqrt{6,987,685.674,965,975,77\ldots.}}{82,26}$, $=$ $.056,441,794,480,744,02 = e$; whence $a + e$, $= z$ the Root, is moſt accurately $12.756,441,794,480,744,02\ldots$ as Dr. *Wallis* found in the fore-mentioned Place; where it may be obſerv'd, that the repetition of the *Calculus* does ever triple the true Figures in the aſſumed a, which the firſt correction, or $\frac{\frac{1}{2}ue^3 - \frac{1}{4}e^4}{\sqrt{\frac{1}{4}ss - bt}}$ does quintuple; which is alſo commodiouſly done by the *Logarithms*. But the other Correction, after the firſt, does alſo double the number of Figures, ſo that it renders the *aſſumed* altogether Seven-fold; yet the firſt Correction is abundantly ſufficient for Arithmetical uſes, for the moſt part.

But, as to what is ſaid concerning the number of Figures rightly taken in the Root, I would have it underſtood ſo that, when a is but $\frac{1}{10}$ part diſtant from the true Root, then the firſt Figure is rightly aſſumed; if it be within $\frac{1}{100}$ part, then the two firſt Figures are rightly aſſumed; and, if within $\frac{1}{1000}$ part, then the three firſt are ſo; which conſequently, managed according to our Rule, do preſently become nine Figures.

It remains now that I add ſomething concerning our *Rational Formula*, viz. $e = \frac{sb}{ss \pm tb}$, which ſeems expeditious enough, and is not much inferiour to the former, ſince it will triple the given Number of Figures. Now, when we ſhall have formed an Equation from $a \pm e = z$, as before, it will preſently appear, whether a be taken greater or leſs than the Truth; ſince se ought always to have a Sign contrary to the Sign of the difference of the *Reſolvend* and its *Homogeneum Comparationis*, produced from a. Then, ſuppoſing $+ b + se + a - tee = o$, the Diviſor is $ss - tb$, as often as t and b have the ſame Signs; but it is $ss + bt$, when they have different ones. But it ſeems moſt commodious for Practice to write the Theorem thus, $e = \frac{b}{s \pm \frac{tb}{s}}$; ſince in this way the thing is done by one Multiplication and two Diviſions, which otherwiſe would require three Multiplications and one Diviſion.

Let us take now one Example of this Method, from the Root (of the fore-mention'd Equation) 12.7 . . . . , where

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$$\begin{array}{rrrrr} 298.6559 & - 5296.132\,e & + 82.26\,ee & + 29.2\,e^3 & \\ + b & - s & + t & + u & \end{array}$$

$- e^4$ is $= o$, and ſo $\frac{b}{s - \frac{tb}{s}}$ is $= e$; that is, let us make the following Proportion, to wit, as s is to t, ſo is b to $\frac{tb}{s}$, or as 5296.132 is to 82.26, ſo is 298.6559 to $\frac{tb}{s}$; which will therefore be $(= \frac{82.26 \times 298.6559}{5296.132})$ $= 4.638,75$; and conſequently the Diviſor $s - \frac{tb}{s}$ will be $(= 5296.132 - 4.638,75) = 5291.493,25$, and $\frac{b}{s - \frac{tb}{s}}$ will be $(= \frac{298.6559}{5291.493,25}) = 0.056,441$; that is, e will be $= 0.056,441$, in which all the Figures are true. And conſequently z, or $a + e$, or the Root of the propoſed Equation $z^4 - 80z^3 + 1998z^2 - 14,937z + 5000 = o$, will be $(= 12.7 + 0.056,441) = 12.756,441$; of which all the eight Figures are true. But this *Rational Formula* $e = \frac{sb}{ss \pm tb}$, or $\frac{b}{s \pm \frac{tb}{s}}$, cannot be corrected, as the foregoing *Irrational* one was; and ſo, if more Figures of the Root are deſired, 'tis the beſt to make a new Suppoſition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly ſatisfie even the moſt ſcrupulous.

AN APPENDIX

TO

DR. EDMUND HALLEY'S TRACT ON THE RESOLUTION OF ALGEBRÄICK EQUATIONS OF ALL DEGREES BY APPROXIMATION.

An APPENDIX *to the foregoing Tract of Dr.* EDMUND HALLEY, *intitled, "A New, Exact, and Eaſy, Method of finding the Roots of Equations Generally, and that without any previous Reduction."*

By FRANCIS MASERES, Esq.

CURSITOR BARON OF THE COURT OF EXCHEQUER IN ENGLAND.

Article 1. DR. HALLEY conſidered his method of reſolving affected equations by approximation (which is deſcribed in the foregoing tract,) as an improvement upon that which had been given by Mr. Joſeph Raphſon in his valuable treatiſe, intitled, *Analyſis Æquationum Univerſalis*, which had been publiſhed in the year 1690; becauſe the new figures in the value of the root ſought that are diſcovered by every proceſs of his approximation, were twice as many as the new figures in the ſaid value diſcovered by the correſponding proceſs of Mr. Raphſon's approximation. But this circumſtance of quicker diſpatch, or rather of the appearance of it, (ſince the reality of it depends upon the facility and number of the operations to be performed in each

each proceſs of the approximation as well as on the number of the ſaid proceſſes,) is not the only thing to be taken into conſideration in order to determine which of the two methods is the moſt convenient: but the ſimplicity and perſpicuity of the reaſonings employed in them;—and the greater or leſs danger of falling into miſtakes, or difficulties, ariſing from having more Algebraick quantities to arrange and manage in Dr. Halley's method than in Mr. Raphſon's;—and ſometimes having two roots, or, in the language of modern Algebraiſts, two real and affirmative roots, to examine and reaſon upon in the proceſſes of the former method, and never more than one ſuch root in thoſe of the latter;—are, all of them, circumſtances of great importance in forming a judgement on this queſtion. And Mr. Raphſon always continued to prefer his own method of approximation, (which proceeded by the repeated reſolutions of mere ſimple equations,) to that of Dr. Halley, (which proceeded by the reſolution of quadratick equations,) after the publication of the foregoing tract of Dr. Halley as well as before it. For he informs us in the ſecond edition of his *Analyſis Æquationum Univerſalis*, (which was publiſhed in the year 1697) that he himſelf had had, at the time of preparing that treatiſe for it's firſt publication in the year 1690, the very ſame thought as Dr. Halley, of retaining the terms of the ſecond, or transformed, equation that involve the ſquare of the unknown root of the ſaid equation as well as the terms that involve the ſimple power of the ſaid root, and ſo reſolving a quadratick equation inſtead of a ſimple one; but that, after full conſideration of the matter, he had rejected that method

of

of proceeding as liable to more perplexity than the other, and had reſolved to proceed only by the reſolution of ſimple equations. And Sir Iſaac Newton, in his method of reſolving equations by approximation, (which differs a little from Mr. Raphſon's,) proceeds only by the reſolution of ſimple equations, except in a few particular caſes, in which it may ſometimes be convenient to proceed by the reſolution of a quadratick equation, as, for example, in performing the firſt proceſs of approximation to the true value of x, when the firſt near value of it, obtained by conjecture, or otherwiſe, and which is taken for the baſis of the approximation, is not ſufficiently near the truth. And, as far as my experience in theſe calculations has enabled me to form a judgement on the ſubject, I am inclined to join with *Mr. Raphſon* in preferring his method of proceeding by the reſolution of only ſimple equations, to this of *Dr. Halley* by the reſolution of quadraticks. But, that the reader may be the better able to form a judgement for himſelf upon this ſubject, I ſhall now proceed to give a ſhort view of theſe two different methods of approximating to the values of the roots of equations, and to point out, firſt, the circumſtances in which they agree, and ſecondly, thoſe in which they differ; and afterwards I ſhall proceed to conſider the three equations $x^4 - 3x^2 + 75x = 10{,}000$, and $x^3 - 17x^2 + 54x = 350$, and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, (which Dr. Halley has choſen in the foregoing tract for the examples of his method of approximation,) and ſhall reſolve them, firſt, by Dr. Halley's, and afterwards by Mr. Raphſon's method of approximation.

Art. 2.

Art. 2. In both theſe methods of reſolving equations by approximation we muſt begin by finding by conjecture, or otherwiſe, in the beſt manner we can, a firſt near value of x, or the root ſought, which firſt near value Dr. Halley denotes by the letter a. We muſt then ſubſtitute ſuch near value inſtead of x in the compound quantity which forms the left-hand ſide of the propoſed equation, and which contains all the terms that involve the unknown quantity x. Thus, if the propoſed equation is the cubick equation $x^3 + px^2 + qx = r$, and a is found, by conjecture or otherwiſe, to be nearly equal to x, we muſt ſubſtitute a inſtead of x in the compound quantity $x^3 + px^2 + qx$, which will thereby be converted into the compound quantity $a^3 + pa^2 + qa$, which conſiſts intirely of known terms. We muſt then compare this known quantity $a^3 + pa^2 + qa$, reſulting from this ſubſtitution, with r, the abſolute term, or known quantity, of the propoſed equation $x^3 + px^2 + qx = r$: and, if the ſaid reſult is leſs than the ſaid abſolute term, we may conclude that the ſaid near value, a, of the root x of the ſaid equation is leſs than it's true value; and, if the ſaid reſult is greater than the ſaid abſolute term, we may conclude that the ſaid near value is greater than the true value of x. It muſt, however, be obſerved that this concluſion will not always be juſt, becauſe in ſome equations that have more than one root, or in the language of modern Algebraiſts, more than one real and affirmative root, it ſometimes happens that, while the unknown quantity x increaſes, the compound quantity that forms the left-hand ſide of the propoſed equation will decreaſe. Thus, if the propoſed equation had been

x^3

$x^3 - px^2 + qx = r$, inſtead of $x^3 + px^2 + qx = r$, and the magnitudes of the two co-efficients p and q and the abſolute term r had been ſuch that the equation admitted of three real and affirmative roots, and a had been a near value of the middlemoſt of the ſaid three roots, the compound quantity $x^3 - px^2 + qx$ would have been greater than the abſolute term r, if a was leſs than the true value of the middle root of the equation; and, if a was greater than the ſaid true value, the ſaid compound quantity would have been leſs than r. But theſe caſes happen but rarely: and, when they do happen, we muſt make uſe of ſome further reaſonings concerning the roots of the equation and the limits of their magnitudes, in order to determine whether a, or the near value of x which we have choſen for the baſis of our approximation, is greater, or leſs, than it's true value.

Art. 3. When it is determined in the foregoing manner, or by ſome further reaſonings ſuited to the nature of the inquiry, that a is leſs than the true value of x, or the root ſought, we muſt then begin our further approximation to it's true value by ſuppoſing the unknown exceſs of x above a, it's firſt near value, to be denoted by ſome other letter of the alphabet and to be added to a. Dr. Halley on this occaſion makes uſe of the letter e: and we will therefore employ the ſame letter. And then we ſhall have $a + e = x$. We muſt next ſubſtitute this binomial quantity $a + e$ inſtead of x in the propoſed equation. And, if this equation be the cubick equation which was juſt now taken as an example of theſe methods of approximation, to wit, the equation $x^3 + px^2 + qx = r$, we ſhall, by means of this ſubſtitution,

ſubſtitution, convert, or transform, the ſaid equation into the following equation, to wit, $\overline{a+e}^3 + p \times \overline{a+e}^2 + q \times \overline{a+e} = r$, or $a^3 + 3a^2e + 3ae^2 + e^3 + p \times \overline{aa + 2ae + ee} + q \times \overline{a+e} = r$, or

$$\left\{ \begin{array}{r} a^3 + 3a^2e + 3ae^2 + e^3 \\ + pa^2 + 2pae + pe^2 \\ + qa + qe \end{array} \right\} = r,$$

in which the unknown quantity that remains to be diſcovered is e, or the difference between x and a, inſtead of the original unknown quantity x itſelf. Now this quantity e is leſs than $a + e$, or x, and uſually much leſs than it, namely, only a tenth part of it, or ſome leſs part. And conſequently, as a is nearly equal to x, e will be alſo much leſs than a, and, *à fortiori*, leſs than $3a$, and, ſtill more, leſs than $3a + p$. Therefore e^3 (which is equal to $e \times e^2$) will be much leſs than $\overline{3a + p} \times e^2$, or than $3ae^2 + pe^2$, and, *à fortiori*, leſs than $\overline{3a^2 + 2pa + q} \times e$, or than $3a^2e + 2pae + qe$. And conſequently, if e^3 be taken away from the left-hand ſide of the equation, the remaining terms of the ſaid left-hand ſide of the equation, to wit, the terms

$$\left\{ \begin{array}{r} a^3 + 3a^2e + 3ae^2 \\ + pa^2 + 2pae + pe^2 \\ + qa + qe \end{array} \right\},$$

will ſtill be very nearly equal to the abſolute term r of the firſt equation. But this curtailed equation is only a quadratick equation, and may therefore be reſolved by the

the known methods of reſolving ſuch equations; which methods are eaſy and accurate. Therefore Dr. Halley adviſes that, in order to determine the value of e, or the difference between a and x, we ſhould reject only the little quantity e^3, or the cube of e, from the foregoing transformed equation, and ſhould reſolve the remaining quadratick equation by the common method of reſolving ſuch equations. And for this purpoſe he directs us to put s for the compound quantity $3a^2 + 2pa + q$, which is the co-efficient of the unknown quantity e in the ſaid quadratick equation, and t for the compound quantity $3a + p$, which is the co-efficient of ee, or of the ſquare of the unknown quantity e in the ſame equation; after which abbreviations of the notation, the equation will be as follows, to wit, $a^3 + pa^2 + qa + se + te^2 = r$. Therefore, (ſubtracting $a^3 + pa^2 + qa$ from both ſides,) we ſhall have $se + te^2 = r - a^3 - pa^2 - qa$, and (dividing both ſides by t,) we ſhall have $\frac{se}{t} + e^2 = \frac{r - a^3 - pa^2 - qa}{t}$, or (putting b for $r - a^3 - pa^2 - qa$) $\frac{se}{t} + e^2 = \frac{b}{t}$. Therefore $e^2 + \frac{se}{t} + \frac{ss}{4tt}$ will be $(= \frac{b}{t} + \frac{ss}{4tt} = \frac{4bt}{4tt} + \frac{ss}{4tt}) = \frac{4bt + ss}{4tt}$, and $e + \frac{s}{2t}$ will be $= \frac{\sqrt{4bt + ss}}{2t}$, and e will be $= \frac{\sqrt{4bt + ss} - s}{2t}$.

Q. E. I.

Therefore

Therefore $a + \frac{\sqrt{4bt + ss} - s}{2t}$ will be a near value of $a + e$, or x, or a ſecond near value of x, which will approach much nearer to it's true value than a, it's firſt near value, did. And, in general, it will be found that the number of decimal figures in this ſecond near value of x that are exact, or agree with the figures of it's true value, is nearly three times the number of the figures that are exact in a, or it's firſt near value: or, to expreſs this matter more preciſely, it will be found that, if a, the firſt near value of x, agrees with it's true value in three, or four, or five figures, or, in general, in n figures, the ſecond near value of x, to wit, $a + \frac{\sqrt{4bt + ss} - s}{2t}$, will, accordingly, agree with the ſaid true value in $3 \times 3 - 1$, or $9 - 1$, or 8, figures, or in $4 \times 3 - 1$, or $12 - 1$, or 11, figures, or in $5 \times 3 - 1$, or $15 - 1$, or 14, figures, or, in general, in $3n - 1$ figures, and in the moſt unfavourable caſes in $3n - 2$ figures. This is certainly a great degree of exactneſs to be attained by a ſingle proceſs of approximation from a, the firſt near value of x, (obtained by conjecture, or otherwiſe,) towards it's true value; and it conſtitutes the great merit of this method of approximation by reſolving a quadratick equation, which Dr. Halley has recommended.

Art. 4. But Mr. Raphſon has thought proper to reject this method of approximation, notwithſtanding this advantage, and to carry on his approximation, from a, the firſt near value of it, towards $a + e$, or it's true value, by

by reſolving only a ſimple equation. His method of proceeding is as follows. When he has ſubſtituted $a + e$ inſtead of x in the propoſed equation $x^3 + px^2 + qx = r$, and thereby transformed the ſaid equation into the equation

$$\left\{\begin{array}{rrrrr} & a^3 & + 3a^2e & + 3ae^2 & + e^3 \\ + & pa^2 & + 2pae & + pe^2 & \\ + & qa & + qe & & \end{array}\right\} = r,$$

he rejects the two terms $3ae^2$ and pe^2, which involve the ſquare of the unknown quantity e, as well as the term e^3, or the cube of the ſaid unknown quantity, becauſe they are, all of them, much ſmaller than the terms $3a^2e + 2pae + qe$, which involve the ſimple power of e; and he thereby reduces the ſaid transformed equation from a cubick equation to a mere ſimple equation, to wit, the equation

$$\left\{\begin{array}{rrrr} & a^3 & + 3a^2e & \\ + & pa^2 & + 2pae & \\ + & qa & + qe & \end{array}\right\} = r,$$

or $a^3 + pa^2 + qa + \overline{3a^2 + 2pa + q} \times e = r$; the reſolution of which is ſtill more ſimple and eaſy than that of a quadratick equation. For we need only ſubtract $a^3 + pa^2 + qa$ from both ſides of the equation, by which it becomes $\overline{3a^2 + 2pa + q} \times e = r - a^3 - pa^2 - qa$; or, if (agreeably to Dr. Halley's notation,) we put s for the compound quantity $3a^2 + 2pa + q$ (which is the co-efficient of e,) and b for the abſolute

ſolute term $r - a^3 - pa^2 - qa$, it will become $se = b$; and conſequently e will be $= \frac{b}{s}$. Q. E. I.

Therefore $a + \frac{b}{s}$ will be a near value of $a + e$, or a ſecond near value of x, which will approach much nearer to it's true value than a it's firſt near value did, but not ſo near as the former expreſſion $a + \frac{\sqrt{4bt + ss} - s}{2t}$, which was obtained, in the foregoing method recommended by Dr. Halley, by reſolving a quadratick equation. For the number of figures in this ſecond near value, $a + \frac{b}{s}$, of the unknown quantity x, that are exact, or agree with the figures of it's true value, will be not nearly triple, but only nearly double, of the number of figures that are exact in a, or it's firſt near value; or, to expreſs this degree of approximation more preciſely, if the number of figures that are exact in a, or the firſt near value of x, is three, or four, or five, or, in general, n, figures, the number of figures that will be exact in $a + \frac{b}{s}$, or the ſecond near value of x, will, accordingly, be $3 \times 2 - 1$, or $6 - 1$, or 5 figures; or $4 \times 2 - 1$, or $8 - 1$, or 7, figures, or $5 \times 2 - 1$, or $10 - 1$, or 9, figures, or, in general, $2n - 1$ figures, and in the moſt unfavourable caſes will be $2n - 2$ figures. This is a very conſiderable degree of exactneſs to be attained by a ſingle proceſs of approximation

approximation performed by the reſolution of a mere ſimple equation, though it is not ſo great as the degree of exactneſs attained by means of the former expreſſion $a + \frac{\sqrt{4bt + ss} - s}{2t}$, which is derived from the more difficult operation of reſolving a quadratick equation.

Art. 5. And, if the approximation to the value of x by Mr. Raphſon's method be carried one ſtep further, by putting c for the quantity $a + \frac{b}{s}$, or the ſecond near value of x, and f for the unknown difference between c and x, and ſubſtituting $c - f$ (which will be equal to x,) inſtead of x in the propoſed equation $x^3 + px^2 + qx = r$, and reſolving the transformed equation, ariſing from ſuch ſubſtitution, as if it were a mere ſimple equation, or expunging from it the term f^3, and alſo the two terms that will involve the ſquare of f, the value of the reſidual quantity $c - f$ that will be thereby obtained, or the 3d near value of x, will agree with it's true value in nearly twice as many figures as were exact in the 2nd near value c, or $a + \frac{b}{s}$; or, if we put m for the number of figures that were exact in that ſecond near value, (which was $2n - 1$,) the number of figures that will be exact in the value of $c - f$, or the 3d near value of x, will be $2m - 1$, (or $2 \times \overline{2n - 1} - 1$, or $4n - 2 - 1$,) or $4n - 3$, which, if n is any number

ber greater than 2, will be greater than $3n - 1$, or the number of figures that are exact in the expreſſion $a + \frac{\sqrt{4bt + ss} - s}{2t}$, or the ſecond near value of x obtained by Dr. Halley's method of proceeding. So that, when n, or the number of figures in a, or the firſt near value of x, that are exact, or agree with the figures of it's true value, is greater than 2, the degree of exactneſs that will be attained by two ſteps, or proceſſes, of Mr. Raphſon's method of approximation will be greater than that which is attained by one ſtep, or proceſs, of Dr. Halley's method.

Art. 6. Having now deſcribed theſe two methods of reſolving high equations by approximation which are recommended by theſe two celebrated mathematicians, and having ſhewn how far they co-incide with each other, and in what circumſtances they differ, I ſhall now proceed to illuſtrate the ſubject further in a practical way, by reſolving the three equations which Dr. Halley has choſen in the foregoing Tract as examples of his method of reſolution, to wit, the cubick equation $x^3 - 17xx + 54x = 350$, and the two biquadratick equations $x^4 - 3x^2 + 75x = 10{,}000$ and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, by both theſe methods, beginning, in each of theſe examples, with the method of Dr. Halley. And, to the end that the reader may be the better able to compare the two methods with each other, and form a judgement of the ſeveral advantages or inconveniencies that belong to each of them, I ſhall, in

in reſolving each of theſe equations by both methods, begin the approximations to the ſecond near values of them (which have been above denoted by $a + e$,) from the ſame firſt near value a, and ſhall carry the approximations by both methods to nearly the ſame degree of exactneſs: and I ſhall alſo ſet forth the grounds and reaſons of the conjectures which I ſhall make concerning the ſaid firſt near values denoted by a; which has not been done by Dr. Halley in the foregoing Tract: and I ſhall ſometimes make choice of a different firſt near value a from that which Dr. Halley has adopted, if I ſhall find that I am able to obtain one that will be nearer to the truth than that which he has choſen, and to ſupport the conjecture to which I give the preference by good and obvious reaſons.

EXAMPLES OF THE RESOLUTIONS OF HIGH EQUATIONS BY BOTH DR. HALLEY'S AND MR. RAPHSON'S METHODS OF APPROXIMATION.

EXAMPLE I.—A CUBICK EQUATION.

Art. 7. Let it be required to reſolve the cubick equation $x^3 - 17xx + 54x = 350$ by Dr. Halley's method of approximation.

We muſt firſt endeavour to find a tolerably near value of x, which ſhall be true to at leaſt one place of figures, by a conjecture grounded on the obvious properties of this equation. Now ſuch a near value may be found in the manner following.

In this equation $x^3 - 17xx + 54x = 350$ let us, firſt, ſuppoſe x^3 to be equal to $17xx$, and conſequently x to be $= 17$. Then will the compound quantity $x^3 - 17xx + 54x$ be $= 54x = 54 \times 17 = 918$; which is much greater than the abſolute term, 350, of the equation. Therefore the true value of x in this equation muſt be leſs than 17.

We will therefore ſuppoſe, in the ſecond place, that x (which now appears to be conſiderably leſs than 17,) is $= 14$, and ſubſtitute 14 inſtead of it in the compound quantity $x^3 - 17xx + 54x$.

Now, if x be ſuppoſed $= 14$, we ſhall have $xx = 196$, and $x^3 = 2744$, and $17xx\ (= 17 \times 196) = 3332$, and $54x\ (= 54 \times 14) = 756$, and conſequently $x^3 - 17xx + 54x\ (= 2744 - 3332 + 756 = 3500 - 3332) = 168$; which is ſomething leſs than half the abſolute term 350. We will therefore make a third conjecture, and ſuppoſe x to be $= 15$, and try the effect of the ſubſtitution of this number inſtead of x in the compound quantity $x^3 - 17xx + 54x$. Now, if x be ſuppoſed to be $= 15$, we ſhall have $xx = 225$, and $x^3 = 3375$, and $17xx\ (= 17 \times 225) = 3825$,

and

and $54x$ $(= 54 \times 15) = 810$, and conſequently $x^3 - 17xx + 54x$ $(= 3375 - 3825 + 810 = 4185 - 3825) = 360$; which is a little greater than the abſolute term 350 of the propoſed equation $x^3 - 17xx + 54x = 350$. We may therefore now conclude with certainty that the true value of x in the ſaid equation muſt be greater than 14 and leſs than 15; ſince, while x increaſes from 14 to 15, the compound quantity $x^3 - 17xx + 54x$ increaſes from 168 to 360. And the ſaid true value of x will evidently be much nearer to 15 than to 14, becauſe the number 360, to which the ſaid compound quantity is equal when x is equal to 15, approaches much nearer to the number 350, or the abſolute term of the propoſed equation, than the number 168, which is the value of the ſaid compound quantity when x is $= 14$. We will therefore take the number 15 for a, or the firſt near value of x in the propoſed equation $x^3 - 17xx + 54x = 350$.

Art. 8. Having thus pitched upon the number 15 for a, or the firſt near value of x, let e be put for the unknown exceſs of 15, or a, (which we know to be greater than x) above x. Then we ſhall have $x = a - e$, and x^2 $(= \overline{a-e}^2) = aa - 2ae + ee$, and x^3 $(= \overline{a-e}^3) = a^3 - 3a^2e + 3ae^2 - e^3$, and $17x^2$ $(= 17 \times \overline{aa - 2ae + ee}) = 17aa - 34ae + 17ee$, and $54x$ $(= 54 \times \overline{a - e}) = 54a - 54e$. Therefore $x^3 - 17xx + 54x$ will be =

$$\left\{\begin{array}{r} a^3 - 3a^2e + 3ae^2 - e^3 \\ - 17aa + 34ae - 17e^2 \\ + 54a - 54e. \end{array}\right\}$$

But $x^3 - 17xx + 54x$ is $= 350$.

Therefore

$$\left\{\begin{array}{r} a^3 - 3a^2e + 3ae^2 - e^3 \\ - 17aa + 34ae - 17e^2 \\ + 54a - 54e \end{array}\right\}$$

will alſo be $= 350$,

But $a^3 - 17aa + 54a$ has been ſhewn to be $= 360$; and $3a^2$ is $(= 3 \times \overline{15}^2 = 3 \times 225) = 675$, and $34a$ is $(= 34 \times 15) = 510$, and $3a$ is $(= 3 \times 15) = 45$.

Therefore we ſhall have $360 - 675e + 510e - 54e + 45e^2 - 17e^2 - e^3 = 350$, or $360 - 219e + 28e^2 - e^3 = 350$, and (adding $219e + e^3$ to both ſides,) $360 + 28e^2 = 350 + 219e + e^3$, and (ſubtracting 350 from both ſides) $10 + 28e^2 = 219e + e^3$, and (ſubtracting $28e^2$ from both ſides,) $219e - 28e^2 + e^3 = 10$, and (omitting e^3 on account of it's ſmallneſs,) $219e - 28e^2$ nearly $= 10$. Therefore (dividing all the terms by 28) we ſhall have $\frac{219e}{28} - e^2 = \frac{10}{28}$, and (ſubtracting both ſides from $\overline{\frac{219}{56}}^2$) we ſhall have $\overline{\frac{219}{56}}^2 - \frac{219e}{28} + ee$

$+ ee = \overline{\frac{219}{56}}^2 - \frac{10}{28}$ $(= \overline{\frac{219}{56}}^2 - \frac{20}{56} = \frac{\overline{219}^2}{\overline{56}^2}$ $- \frac{20 \times 56}{\overline{56}^2} = \frac{\overline{219}^2 - 20 \times 56}{\overline{56}^2} = \frac{47{,}961 - 1120}{\overline{56}^2})$ $= \frac{46{,}841}{\overline{56}^2}$. Therefore the ſquare-root of the trinomial quantity $\overline{\frac{219}{56}}^2 - \frac{219e}{28} + ee$ will be $= \frac{\sqrt{46{,}841}}{56}$ $= \frac{216.427{,}817, \&c}{56}$. But this trinomial quantity has two ſquare-roots, to wit, $\frac{219}{56} - e$ and $e - \frac{219}{56}$, in the former of which the unknown quantity e is leſs, and in the latter of which it is greater, than $\frac{219}{56}$, or 3.9107. Now the latter of theſe values of e, being greater than 3.9107, will be greater than the value of e which we are ſeeking, or than the exceſs of 15 above x, becauſe x has been ſhewn to be greater than 14, which differs from 15 by only 1. It follows therefore that $\frac{219}{56} - e$, or the former of the two ſquare-roots of the trinomial quantity $\overline{\frac{219}{56}}^2 - \frac{219e}{28} + e^2$, will be that which is to be uſed on the preſent occaſion. We ſhall therefore have $\frac{219}{56} - e = \frac{216.427{,}817, \&c}{56}$, and (adding e to both ſides,) $\frac{219}{56} = \frac{216.427{,}817, \&c}{56} + e$, and (ſubtracting

$\frac{216.427,817, \&c}{56}$ from both ſides) $e = \frac{219.000,000 - 216.427,817,}{56} = \frac{2.572,183}{56} =$ 0.045,931, or, more nearly, 0.045,932. Therefore $a - e$, or $15 - e$, will be $= 15.000,000 - 0.045,932 = 14.954,068$; that is, the ſecond near value of x, or the root of the propoſed equation $x^3 - 17xx + 54x = 350$, obtained by this firſt proceſs of Dr. Halley's method of reſolving equations by approximation, will be 14.954,068. Q. E. I.

Art. 9. This number 14.954,068, here found for the value of x in the propoſed equation $x^3 - 17xx + 54x = 350$, is exactly the ſame as that found by Dr. Halley in the foregoing Tract. And it agrees with the true value of x at leaſt in the firſt ſix figures 14.9540. For, if we ſuppoſe x to be $= 14.9540$, we ſhall have $xx = 223.622,116,00$, and $x^3 = 3344.045,122,664,000$, and $17xx$ $(= 17 \times 223.622,116,00) = 3801.575,972,00$, and $54x$ $(= 54 \times 14.9540) = 807.5160$, and conſequently $x^3 - 17xx + 54x = 3344.045,122,664,000 - 3801.575,972,00 + 807.5160$ $(= 4151.561,122,664,000 - 3801.575,972,00) = 349.985,150,664,000$; which is ſomewhat leſs than the abſolute term 350. And, if we ſuppoſe x to be $= 14.9541$, we ſhall have $xx = 223.625,106,81$, and $x^3 = 3344.112,209,747,421$, and $17xx$ $(= 17 \times 223.625,106,81) = 3801.626,815,77$, and $54x$ $(= 54 \times 14.9541) = 807.5214$, and conſequently $x^3 - 17xx + 54x$ $(= 3344.112,209,747,421$

— 3801.626,815,77 + 807.5214 = 4151.633,609,747, 421 — 3801.626,815,77) = 350.006,793,977,421; which is ſomewhat greater than the abſolute term 350. Therefore the true value of x muſt be of an intermediate magnitude between 14.9540 and 14.9541, and conſequently the firſt ſix figures of it muſt be 14.9540.

Q. E. D.

The exceſs of 350.006,793,977,421 above the abſolute term 350 is only 0.006,793,977,421, which is leſs than 0.007; but the exceſs of the abſolute term 350 above 349.985,150,664,000 is 0.014,849,336,000, which is more than double the other exceſs. It therefore ſeems reaſonable to ſuppoſe that the true value of x will approach nearer to 14.9541, which produces the ſmaller exceſs, than to 14.9540, which produces the greater exceſs. And it is very probable that the next two figures 68 of the value 14.954,068, obtained by the foregoing approximation, will likewiſe be exact; and Dr. Halley ſeems to have thought they are ſo. And ſo they will be found to be in the courſe of the following reſolution of this equation by Mr. Raphſon's method of approximation.

The Reſolution of the ſame Equation by Mr. Raphſon's Method.

Art. 10. I ſhall now proceed to reſolve the ſame cubick equation $x^3 - 17xx + 54x = 350$, by Mr. Raphſon's method of approximation, beginning the approximation from the ſame firſt near value of x, to wit, the number 15, which was choſen in the preceding articles for the baſis of the approximation by Dr. Halley's method.

It

It appears from the reaſonings uſed above in art. 75 that the true value of x in this equation is greater than the number 14, and leſs than the number 15, but is nearer to 15 than to 14. We will therefore take the number 15 for it's firſt near value a, which is to be the baſis of our further approximation to it's true value in the manner preſcribed by Mr. Raphſon; and we will denote the exceſs of a, or 15, above the true value of x by the letter e, as we did in the foregoing reſolution of the equation by Dr. Halley's method. And we ſhall then have $x = a - e$, and conſequently $xx = \overline{a-e}^2 = aa - 2ae + ee$, and $x^3 = \overline{a-e}^3 = a^3 - 3a^2e + 3ae^2 - e^3$, and $17xx\,(= 17 \times \overline{aa - 2ae + ee}) = 17aa - 34ae + 17ee$, and $54x\,(= 54 \times \overline{a-e}) = 54a - 54e$. Therefore the compound quantity $x^3 - 17xx + 54x$ will be equal to the compound quantity

$$\left\{\begin{array}{l} a^3 - 3a^2e + 3ae^2 - e^3 \\ -17a^2 + 34ae - 17e^2 \\ +54a - 54e. \end{array}\right\}$$

But the compound quantity $x^3 - 17xx + 54x$ is $= 350$.

Therefore the compound quantity

$$\left\{\begin{array}{l} a^3 - 3a^2e + 3ae^2 - e^3 \\ -17a^2 + 34ae - 17e^2 \\ +54a - 54e \end{array}\right\}$$

will alſo be $= 350$.

This

This equation is accurately true, and is what Mr. Raphſon, in his excellent treatiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, calls *Theorema Vietæum*, or the Theorem of the great French Algebraiſt *Vieta*, or Monſieur *Viète*, who is often and juſtly ſtyled *The Father of Algebra*, though his works have of late years been but little read. This theorem, or transformed equation ariſing from the ſubſtitution of $a - e$, or $a + e$, inſtead of x in the terms of the original equation $x^3 - 17xx + 54x = 350$, or other propoſed original equation, is the grand foundation of all the methods that have been made uſe of for reſolving equations by approximation, as well as of Vieta's own method ſet forth in his Treatiſe intitled, *De numeroſâ poteſtatum purarum atque adfectarum reſolutione tractatus*, which begins in page 163 and ends in page 228 of Schooten's edition of Vieta's Works publiſhed in the year 1646.

Art. 11. From this equation (which is accurately true) we muſt, according to Mr. Raphſon's method of approximation, expunge the three terms $+ 3ae^2 - 17e^2 - e^3$; and then the equation will be

$$\left\{\begin{array}{r} a^3 - 3a^2e \\ - 17a^2 + 34ae \\ + 54a - 54e \end{array}\right\} \text{ nearly } = 350,$$

or $a^3 - 17a^2 + 54a - 3a^2e + 34ae - 54e =$ (nearly) 350, or (if we ſubſtitute 15 inſtead of a in the terms of this equation) $\overline{15}^3 - 17 \times \overline{15}^2 + 54 \times 15 - 3 \times \overline{15}^2 \times e + 34 \times 15 \times e - 54e = 350$, or

3375

$3375 - 17 \times 225 + 54 \times 15 - 3 \times 225 \times e + 34 \times 15 \times e - 54e = 350$, or $337^{5} - 3825 + 810 - 675 \times e + 510 \times e - 54e = 350$, or $4185 - 3825 - 729 \times e + 510e = 350$, or $360 - 219e = 350$. Therefore (adding $219e$ to both ſides,) we ſhall have $360 = 350 + 219e$, and (ſubtracting 350 from both ſides,) $10 = 219e$, and conſequently (dividing both ſides by 219) $e = \frac{10}{219} = 0.0456$. Therefore $a - e$, or $15 - e$, will be $= 15.0000 - 0.0456 = 14.9544$; that is, the root x of the propoſed equation $x^3 - 17xx + 54x = 350$ will be nearly equal to 14.9544. Q. E. I.

Art. 12. This value of x is exact in the five figures 14.954, which ſeems to be a great degree of exactneſs to be attained by the reſolution of ſo eaſy a ſimple equation as the equation $219e = 10$. And a repetition of this proceſs will give it us exact to four or five figures more, or to nine or ten figures in the whole, as will be apparent from the following operations.

Let us ſuppoſe the firſt five figures 14.954 of the foregoing value of x, to wit, 14.9544, to be exact, as we indeed know them to be by means of the ſubſtitutions made in art. 9; and let us ſubſtitute this number 14.954 inſtead of x in the compound quantity $x^3 - 17xx + 54x$, in order to diſcover whether the value of the ſaid compound quantity ariſing from ſuch ſubſtitution will be greater, or leſs, than the abſolute term 350 of the propoſed equation, as if we had not already made this inquiry

quiry in art. 9, and found the ſaid reſult to be leſs than 350.

Then we ſhall have xx, or $\overline{14.954}|^2$, = 223.622,116, and x^3, or $\overline{14.954}|^3$, = 3344.045,122,664, and $17xx$ (= 17 × 223.622,116) = 3801.575,972, and $54x$ (= 54 × 14.954) = 807.516, and conſequently $x^3 - 17xx - 54x$ (= 3344.045,122,664 − 3801.575,972 + 807.516 = 4151.561,122,664 − 3801.575,972) = 349.985,150,664; which is leſs than 350, or the abſolute term of the propoſed equation $x^3 - 17xx + 54 = 350$. Therefore 14.954 will be leſs than the true value of x in that equation. Let c be put for 14.954, and f for the exceſs of x above 14.954, or c. Then we ſhall have $x = c + f$, and $xx = \overline{c+f}|^2 = cc + 2cf + ff$, and $x^3 = \overline{c+f}|^3 = c^3 + 3c^2f + 3cf^2 + f^3$, and $17xx$ (= 17 × $\overline{cc + 2cf + ff}$) = $17cc + 34cf + 17ff$, and $54x$ (= 54 × $\overline{c+f}$) = $54c + 54f$, and conſequently $x^3 - 17xx + 54x$ =

$$\left\{\begin{array}{r} c^3 + 3c^2f + 3cf^2 + f^3 \\ -17cc - 34cf - 17f^2 \\ +54c + 54f. \end{array}\right\}$$

But $x^3 - 17xx + 54x$ is = 350.

Therefore the compound quantity

$c^3 +$

$$\left\{\begin{array}{r} c^3 + 3c^2f + 3cf^2 + f^3 \\ - 17c^2 - 34cf - 17f^2 \\ + 54c + 54f \end{array}\right\}$$

will alſo be = 350.

Now let the three terms $+ 3cf^2 - 17f^2 + f^3$ (which will evidently be much ſmaller than the terms $+ 3c^2f - 34cf + 54f$,) be expunged from the equation. And we ſhall then have

$$\left\{\begin{array}{r} c^3 + 3c^2f \\ - 17c^2 - 34cf \\ + 54c + 54f \end{array}\right\}$$

= (nearly) 350.

But $c^3 - 17c^2 + 54c$, or $\overline{14.954}^3 - 17 \times \overline{14.954}^2 + 54 \times 14.954$, has been ſhewn to be = 349.985,150,664; and $3c^2 \times f$ is $(= 3 \times \overline{14.954}^2 \times f = 3 \times 223.622,116 \times f) = 670.866,348 \times f$, and $34c \times f$ is $(= 34 \times 14.954 \times f) = 508.436 \times f$, and conſequently $3c^2f - 34cf + 54f$ will be $(= 670.866,348 \times f - 508.436 \times f \times 54 \times f = 724.866,348 \times f - 508.436 \times f) = 216.430,348 \times f$.

Therefore $349.985,150,664 + 216.430,348 \times f$ will be = 350, and conſequently (ſubtracting 349.985,150,664 from both ſides,) $216.430,348 \times f$ will be = 0.014,849,336, and f will be $= \frac{0.014,849,336}{216.430,348} =$ 0.000,068,610,2.

Therefore

Therefore x, or $c + f$, or 14.954 + f, will be = 14.954,068,610,2. Q. E. I.

Art. 13. This number, 14.954,068,610,2, is probably exact in all it's figures; but is certainly ſo in the firſt ten figures 14.954,068,61, of which the firſt eight figures, 14.954,068, are the ſame as thoſe of the former value of x, obtained by Dr. Halley's method of approximation. So that one proceſs of Dr. Halley's method of approximation gives us the value of x in the cubick equation $x^3 - 17xx + 54x = 350$ exact to eight figures by the reſolution of the quadratick equation $219e - 28e^2 = 10$, and two proceſſes of Mr. Raphſon's method of approximation give us the value of the ſame quantity exact to at leaſt ten places of figures by the reſolution of the two ſimple equations $219e = 10$, and $216.430,348 \times f = 0.014,849,336$. The reader muſt now judge for himſelf which of theſe two methods deſerves the preference.

EXAMPLE II.—A BIQUADRATICK EQUATION.

Art. 14. Let it be required to reſolve the biquadratick equation $x^4 - 3x^2 + 75x = 10,000$ by Dr. Halley's method of approximation.

An Inveſtigation, by Conjectures and Trials, of a, *or the firſt near Value of* x.

In order to find a firſt near value of x in this equation, to be the baſis of a ſecond approximation to it's true

 value

value in the method recommended by Dr. Halley, we may proceed as follows:

If x^4 alone were equal to the abſolute term 10,000, x would be exactly equal to 10. Therefore it ſeems reaſonable to conjecture that in the equation $x^4 - 3xx + 75x = 10{,}000$, (in which the other two terms $3xx$ and $75x$ have contrary ſigns prefixed to them, and therefore have leſs effect in increaſing or diminiſhing the magnitude of x^4 than they would have if they were both marked with the ſame ſign, and conſequently will not make the magnitude of the trinomial quantity $x^4 - 3xx + 75x$ be very different from that of x^4 alone,) the value of x will not be very different from 10. We will therefore ſubſtitute 10 inſtead of x in the compound quantity $x^4 - 3xx + 75x$, in order to diſcover whether the value of it reſulting from this ſuppoſition will be nearly equal to the abſolute term 10,000. Now, if x be ſuppoſed to be $= 10$, we ſhall have $xx = 100$, and and $x^4 = 10{,}000$, and $3xx = 300$, and $75x = 750$, and conſequently $x^4 - 3xx + 75x$ $(= 10{,}000 - 300 + 750) = 10{,}450$; which is greater than the abſolute term 10,000, but not in any great degree. We will therefore make a ſecond ſuppoſition not very different from the former, to wit, that x is equal to 9, and try the effect of this ſuppoſition. Now, if x is $= 9$, we ſhall have $xx = 81$, and $x^4 = 6561$, and $3xx = 243$, and $75x$ $(= 75 \times 9) = 675$, and conſequently $x^4 - 3xx + 75x$ $(= 6561 - 243 + 675 = 7236 - 243) = 6993$. Therefore, while x increaſes from 9 to 10, the the compound quantity $x^4 - 3xx + 75x$ will increaſe from 6993 to 10,450, and therefore will, at ſome one

point of time during the ſaid increaſe of x from 9 to 10 be equal to 10,000, or the abſolute term of the propoſed equation $x^4 - 3xx + 75x = 10{,}000$; or there will be ſome quantity greater than 9, but leſs than 10, which, being ſubſtituted inſtead of x in the compound quantity $x^4 - 3xx + 75x$, will make the ſaid quantity be equal to 10,000; that is, in other words, the true value of x in the equation $x^4 - 3xx + 75x = 10{,}000$ will be greater than 9, but leſs than 10. And, as the value of the ſaid compound quantity $x^4 - 3xx + 75x$ reſulting from the ſubſtitution of 10 inſtead of x in it's terms was 10,450, which exceeds 10,000 by only 450, and the value of the ſaid compound quantity reſulting from the ſubſtitution of 9 inſtead of x in it's terms was only 6993, which falls ſhort of 10,000 by the number 3007, which is a much greater quantity than 450, it ſeems reaſonable to conclude that 10 will differ leſs than 9 from the true value of x in the ſaid equation $x^4 - 3xx + 75x = 10{,}000$. And for theſe reaſons we will pitch upon 10 for the value of a, or the firſt near value of x in the propoſed equation, or for the baſis of the approximation which we are now to make towards it's true value in the manner recommended by Dr. Halley.

A more exact Determination of the Value of x *by Dr. Halley's Method of Approximation.*

Art. 15. Now let e be put for the difference between a, or 10, and the true value of x in the propoſed equation.

And we ſhall then have $x = 10 - e$, or $a - e$, and conſequently $xx = \overline{a - e}^2 = aa - 2ae + ee$, and

$x^4 = \overline{a - e}^4 = a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4$, and $3xx\ (= 3 \times \overline{aa - 2ae + ee}) = 3a^2 - 6ae + 3ee$, and $75x\ (= 75 \times \overline{a - e}) = 75a - 75e$. Therefore the whole compound quantity $x^4 - 3xx + 75x$ will be equal to the compound quantity

$$\left\{ \begin{array}{r} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ -3a^2 + 6ae - 3e^2 \\ +75a - 75e. \end{array} \right\}$$

But the compound quantity $x^4 - 3xx + 75x$ is $= 10{,}000$.

Therefore the compound quantity

$$\left\{ \begin{array}{r} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ -3a^2 + 6ae - 3e^2 \\ +75a - 75e \end{array} \right\}$$

will alſo be $= 10{,}000$.

This is the full transformed equation, and is accurately true. But from this equation Dr. Halley directs us to expunge the two terms $4ae^3$ and e^4, as being very ſmall in compariſon of the terms $6a^2e^2$ and $3e^2$, and, *à fortiori*, in compariſon of the terms $4a^3e$, $6ae$, and $75e$, which are to be retained. And by this omiſſion the equation is reduced to the quadratick equation

$$\left\{ \begin{array}{r} a^4 - 4a^3e + 6a^2e^2 \\ -3a^2 + 6ae - 3e^2 \\ +75a - 75e \end{array} \right\} \text{ nearly } = 10{,}000.$$

Art. 16.

Art. 16. Now, ſince a is = 10, we ſhall have $a^4 - 3a^2 + 75a = 10{,}450$, as has been ſhewn in art. 14; and we ſhall have $4a^3$ $(= 4 \times \overline{10}^3 = 4 \times 1000) = 4000$, and $6a$ $(= 6 \times 10) = 60$, and $6a^2$ $(= 6 \times \overline{10}^2 = 6 \times 100) = 600$, and conſequently

$$\left\{\begin{array}{l} a^4 - 4a^3e + 6a^2e^2 \\ -\ 3a^2 + 6ae - 3e^2 \\ +\ 75a - 75e \end{array}\right\} =$$

$$\left\{\begin{array}{rrrr} 10{,}450 & - 4000e & + 600e^2 \\ & + 60e & - 3e^2 \\ & - 75e & \end{array}\right\}$$

$$= 10{,}450 - 4015e + 597e^2.$$

Therefore $10{,}450 - 4015e + 597e^2$ will be nearly = 10,000. And conſequently (adding $4015e$ to both ſides,) we ſhall have $10{,}450 + 597e^2 = 10{,}000 + 4015e$, and (ſubtracting $597e^2$ from both ſides,) $10{,}450 = 10{,}000 + 4015e - 597e^2$, and, laſtly, (ſubtracting 10,000 from both ſides,) $4015e - 597e^2 = 450$. This quadratick equation muſt now be reſolved; which may be done as follows.

Art. 17. Divide all the terms by 597, the co-efficient of e^2. And we ſhall have $6.725{,}293 \times e - e^2 = 0.753{,}769$. Now let both ſides of this equation be ſubtracted from the ſquare of half the number 6.725,293, which is the co-efficient of e, that is, from the ſquare of 3.362,646, or from 11.307,388,121,316; and we ſhall

then have $\overline{3.362,646}^2 - 6.725,293 \times e + e^2$ (= 11.307,388,121,316 − 0.753,769) = 10.553,619,121,316. Therefore the ſquare-root of the trinomial quantity $\overline{3.362,646}^2 - 6.725,293 \times e + e^2$ will be (= $\sqrt{10.553,619,121,316}$) = 3.248,633.

But this trinomial quantity has two ſquare-roots, to wit, the binomial quantity 3.362,646 − e, and the binomial quantity e − 3.362,646, in which latter quantity e is greater than 3.362,646. But we have ſeen above in art. 14, that e, or the difference between 10 and the true value of x, is leſs than 1. Therefore this latter binomial quantity e − 3.362,646 cannot be that which will give us the true value of e, and conſequently we muſt make uſe of the other ſquare-root of the ſaid trinomial quantity, to wit, the binomial quantity 3.362,646 − e. We ſhall therefore have 3.362,646 − e = 3.248,633; and conſequently (adding e to both ſides,) we ſhall have 3.362,646 = 3.248,633 + e, and e (= 3.362,646 − 3.248,633) = 0.114,013, or (neglecting the three laſt places of figures as not exact,) 0.114. Therefore $a - e$, or 10 − e, will be = 10 − 0.114 = 9.886; that is, the ſecond near value of x in the propoſed equation $x^4 - 3xx + 75x = 10,000$ will be 9.886.

Q. E. I.

Art. 18. Now let 9.886 be ſubſtituted inſtead of x in the compound quantity $x^4 - 3xx + 75x$, in order to diſcover whether the value of that quantity reſulting from

from ſuch ſubſtitution will be greater, or leſs, than the abſolute term 10,000, and conſequently whether the ſaid number 9.886 will be greater, or leſs, than the true value of x in the propoſed equation $x^4 - 3xx + 75x = 10{,}000$. This ſubſtitution may be made as follows.

If x is $= 9.886$, we ſhall have xx $(= \overline{9.886}^2) = 97.732{,}996$, and x^3 $(= \overline{9.886}^3) = 966.188{,}398{,}456$, and x^4 $(= \overline{9.886}^4) = 9551.738{,}507{,}136{,}016$, and conſequently $3xx$ $(= 3 \times 97.732{,}996) = 293.198{,}988$, and $75x$ $(= 75 \times 9.886) = 741.450$, and the whole compound quantity $x^4 - 3xx + 75x$ $(= 9551.738{,}507{,}136{,}016 - 293.198{,}988 + 741.450 = 10{,}293.188{,}507{,}136{,}016 - 293.198{,}988) = 9{,}999.989{,}519{,}136{,}016$, which is very nearly equal to, but ſomewhat leſs than, the abſolute term, 10,000, of the propoſed equation $x^4 - 3xx + 75x = 10{,}000$. Therefore the number 9.886 will be very nearly equal to, but ſomewhat leſs than, the true value of x in that equation. Q. E. I.

A ſecond Proceſs of Dr. Halley's Method of Approximation.

Art. 19. Now, in order to obtain the value of x to a greater degree of exactneſs by means of a ſecond proceſs of Dr. Halley's method of approximation, let c be put for the value of x already found, to wit, 9.886, and f for the unknown difference by which the true value of x exceeds 9.886.

Then we ſhall have $x = c + f$, and conſequently xx $(= \overline{c+f}^2 = cc + 2cf + ff$, and x^4 $(= \overline{c+f}^4)$ $= c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4$, and $3xx$ $(=$

$3 \times \overline{cc + 2cf + ff} = 3cc + 6cf + 3ff$, and $75x\ (= 75 \times \overline{c+f}) = 75c + 75f$. Therefore the compound quantity $x^4 - 3xx + 75x$ will be equal to the compound quantity

$$\left\{\begin{array}{r} c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4 \\ -3cc - 6cf - 3f^2 \\ +75c + 75f. \end{array}\right\}$$

Therefore this laſt compound quantity will be = the abſolute term 10,000.

From this equation let the two terms $4cf^3$ and f^4 be expunged, on account of their extreme ſmallneſs in compariſon of the terms that involve ff and f. And we ſhall then have the compound quantity

$$\left\{\begin{array}{r} c^4 + 4c^3f + 6c^2f^2 \\ -3c^2 - 6cf - 3f^2 \\ +75c + 75f \end{array}\right\}, \text{ nearly, } = 10{,}000.$$

But we have ſeen that $c^4 - 3c^2 + 75c$, or $\overline{9.886}^4 - 3 \times \overline{9.886}^2 + 75 \times 9.886$, is = 9,999.989,519,136,016; and $4c^3$ will be $(= 4 \times \overline{9.886}^3 = 4 \times 966.188,398,456) = 3864.753,593,824$, and $6c$ will be $(= 6 \times 9.886) = 59.316$, and conſequently $4c^3 - 6c + 75$ will be $(= 3864.753,593,824 - 59.316 + 75 = 3939.753,593,824 - 59.316)$ 3880.437,593,824; and $6c^2$ will be $(= 6 \times \overline{9.886}^2 = 6 \times 97.732,996) = 586.397,976$, and conſequently $6c^2 - 3$ will be $(= 586.397,976 - 3) = 583.397,976$. Therefore the compound quantity

c^4

$$\left\{\begin{array}{l} c^4 + 4c^3f + 6c^2f^2 \\ -3c^2 - 6cf - 3f^2 \\ +75c + 75f \end{array}\right\}.$$

will be = 9,999.989,519,136,016 + 3880.437,593,824 $\times f$ + 583.397,976 $\times ff$. And consequently this last quantity 9,999.989,519,136,016 + 3880.437,593,824 $\times f$ + 583.397,976 $\times ff$ will be, nearly, = 10,000, and therefore (subtracting 9,999.989,519,136,016 from both sides,) 3880.437,593,824 $\times f$ + 583.397,976 $\times ff$ will be = 0.010,480,863,984; which is a quadratick equation now properly reduced into order, and prepared for resolution. And by resolving this equation we shall obtain the value of f, and consequently of $c + f$, or 9.886 + f, or the third near value of x in the proposed equation $x^4 - 3xx + 75x = 10{,}000$. This quadratick equation may be resolved as follows.

Art. 20. Since 3880.437,593,824 $\times f$ + 583.397,976 $\times ff$ is = 0.010,480,863,984, we shall have $\frac{3880.437,593,824 \times f}{583.397,976} + ff = \frac{0.010,480,863,984}{583.397,976}$, that is, 6.651,441,646 $\times f + ff$ = 0.000,017,965,204,569,033. Now let the square of half the co-efficient of f, that is, of half the number 6.651,441,646, or the square of 3.325,720,823, (which square is = 11.060,418,992,535,797,329,) be added to both sides of this equation. And we shall have $\overline{3.325,720,823}^2$ + 6.651,441,646 $\times f + ff$ (= 11.060,418,992,535,797,329 + 0.000,017,965,204,569,033) = 11.060,436,957,740,366,362. Therefore (extracting the square-roots of both sides,) we shall have 3.325,720,823 + f = 3.325,723,523; and con-

conſequently f will be (= 3.325,723,523 — 3.325,720,823) = 0.000,002,700. Therefore $c + f$, or 9.886 + f, will be (= 9.886 + 0.000,002,700) = 9.886,002,700; that is, the third near value of x in the propoſed equation $x^4 - 3xx + 75x = 10{,}000$, obtained by Dr. Halley's method of approximation, will be 9.886,002,700. Q. E. I.

N. B. Dr. Halley (in the foregoing tract, as it is printed in the *Miſcellanea Curioſa*,) makes this more accurate value of x to be 9.886,260,393,649,5, which, he tells us, ſcarce exceeds the truth by 2 in the laſt figure (ſee above, page 15.) But I conceive that, if this is not owing to ſome errors of the preſs, he muſt have made ſome ſlip in his calculation; becauſe the number here found for x, to wit, 9.886,002,700, (with which Dr. Halley's number agrees in only the firſt four figures 9.886,) will be found to agree in the firſt nine figures 9.886,002,70, with the number that will be found for it in the following articles by Mr. Raphſon's method of approximation.

The Reſolution of the ſame Equation by Mr. Raphſon's Method of Approximation.

Art. 21. I now proceed to reſolve the ſame biquadratick equation $x^4 - 3xx + 75x = 10{,}000$ by Mr. Raphſon's method of approximation, beginning the approximation from the ſame firſt near value of x, to wit, the number 10, which was choſen in the foregoing articles for the baſis of the approximation by Dr. Halley's method.

Having found, by the reaſonings uſed in art. 14, that the value of x in the propoſed equation $x^4 - 3xx + 75x = 10{,}000$ will be nearly equal to, but ſomewhat leſs

leſs than, the number 10, let a be $= 10$, and let e denote the unknown difference by which 10 exceeds the true value of x.

Then will x be $= 10 - e$, or $a - e$; and conſequently xx will be $(= \overline{a - e}^2) = aa - 2ae + ee$, and x^3 will be $(= \overline{a - e}^3) = a^3 - 3a^2e + 3ae^2 - e^3$, and x^4 will be $(= \overline{a - e}^4) = a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4$. Therefore $3xx$ will be $= 3 \times \overline{aa - 2ae + ee} = 3aa - 6ae + 3ee$, and $75x$ will be $(= 75 \times \overline{a - e}) = 75a - 75e$. Therefore the compound quantity $x^4 - 3xx + 75x$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} a^4 - 4a^3e + 6a^2e^2 - 4ae^3 + e^4 \\ - 3a^2 + 6ae - 3e^2 \\ + 75a - 75e. \end{array} \right\}$$

Therefore this laſt compound quantity will be $=$ the abſolute term 10,000.

Now let the four terms $+ 6a^2e^2 - 3e^2 - 4ae^3 + e^4$ be expunged from this equation, as being much ſmaller than the three preceeding terms $4a^3e$, $6ae$, and $75e$, which involve the ſimple power of e. And the remaining compound quantity

$$\left\{ \begin{array}{l} a^4 - 4a^3e \\ - 3a^2 + 6ae \\ + 75a - 75e \end{array} \right\}$$

will be nearly $=$ the abſolute term 10,000.

But $a^4 - 3a^2 + 75a$ has been ſhewn in art. 14 to be $= 10{,}450$; and $4a^3$ is $(= 4 \times \overline{10}^3 = 4 \times 1000)$ $= 4000$, and $6a$ is $(= 6 \times 10) = 60$,

Therefore

Therefore $a^4 - 3a^2 + 75a - 4a^3e + 6ae - 75e$ will be $(= 10{,}450 - 4000e + 60e - 75e = 10{,}450 - 4075e + 60e) = 10{,}450 - 4015e$; and conſequently $10{,}450 - 4015e$ will be, nearly, = the abſolute term 10,000. Therefore, (adding $4015e$ to both ſides,) we ſhall have $10{,}000 + 4015e = 10{,}450$, and (ſubtracting 10,000 from both ſides,) $4015e = 450$. Therefore e will be $= \frac{450}{4015} = 0.112$. Therefore $a - e$, or $10 - e$, will be $= 10 - 0.112 = 9.888$; that is, the ſecond near value of x, or the root of the propoſed equation $x^4 - 3xx + 75x = 10{,}000$, will be 9.888.

Q. E. I.

Art. 22. Of this number, 9.888, the firſt three figures, 9.88, are exact, or the ſame with the three firſt figures of the true value of x, and the fourth figure 8 is only a little greater than the truth, the exact fourth figure being a 6. And the reſolution of the ſimple equation $4015e = 450$, by which this value of e has been obtained, is much eaſier and ſhorter than the reſolution of the quadratick equation $4015e - 597ee = 450$, given above in art. 17, by which we obtained the number 9.886 for the ſecond near value of x, by Dr. Halley's method.

Art. 23. Having thus obtained 9.888 for a ſecond near value of x in the propoſed equation $x^4 - 3xx + 75x = 10{,}000$, we will try the degree of it's approximation to the truth by ſubſtituting it inſtead of x in the compound quantity $x^4 - 3xx + 75x$.

Now, if x is $= 9.888$, we ſhall have $xx\ (= \overline{9.888}^2) = 97.772{,}544$, and $x^3\ (= \overline{9.888}^3) = 966.774{,}915{,}072$, and

and x^4 $(= \overline{9.888}^4)$ = 9559.470,360,231,936, and consequently $3xx$ $(= 3 \times 97.772,544)$ = 293.317,632, and $75x$ $(= 75 \times 9.888)$ = 741.600, and $x^4 + 75x$ $(= 9559.470,360,231,936 + 741.600)$ = 10,301.070,360,231,936, and $x^4 - 3xx + 75x$ $(= 10,301.070,360,231,936 - 293.317,632)$ = 10,007.752,728,231,936; which is a little greater than 10,000, or the abſolute term of the equation $x^4 - 3xx + 75x = 10,000$. Therefore 9.888 will be a little greater than the true value of x in that equation.

A ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Art. 24. In order to obtain a nearer value of x, let f be put for the exceſs of 9.888 above the true value of x. And we ſhall then have $x = 9.888 - f$, and conſequently xx $(= \overline{9.888 - f}^2 = \overline{9.888}^2 - 2 \times 9.888 \times f +$ &c) = 97.772,544 $-$ 19.776 $\times f +$ &c, and x^4 $(= \overline{9.888 - f}^4 = \overline{9.888}^4 - 4 \times \overline{9.888}^3 \times f +$ &c = 9559.470,360,231,936 $-$ 4 $\times$ 966.774,915,072 $\times f +$ &c) = 9559.470,360,231,936 $-$ 3867.099,660,288 $\times f +$ &c, and $3xx$ $(= 3 \times \overline{97.772,544 - 19.776 \times f +}$ &c = 3 $\times$ 97.772,544 $-$ 3 $\times$ 19.776 $\times f$) = 293.317,632 $-$ 59.328 $\times f +$ &c, and $75x$ $(= 75 \times \overline{9.888 - f} = 75 \times 9.888 - 75 \times f) = 741.600 - 75f$.

Therefore the compound quantity $x^4 - 3xx + 75x$ will be = the compound quantity

9559.

$$\left\{\begin{array}{llll} 9559.470,360,231,936 & - & 3867.099,660,288 & \times f + \&c \\ -293.317,632, & + & 59.328 & \times f - \&c \\ +741.600 & - & 75 \times f & \end{array}\right\}$$

$= 10,007.752,728,231,936 - 3882.771,660,288 \times f$ &c.

But the compound quantity $x^4 - 3xx + 75x$ is $= 10,000$.

Therefore we ſhall have $10,007.752,728,231,936 - 3882.771,660,288 \times f$ &c alſo $= 10,000$; and, (adding $3882.771,660,288 \times f$ to both ſides,) $10,007.752,728,231,936$ nearly $= 10,000 + 3882.771,660,288 \times f$, and (ſubtracting 10,000 from both ſides,) $3882.771,660,288 \times f =$ nearly, $7.752,728,231,936$,

and $f = \frac{7.752,728,231,936}{3882.771,660,288} = 0.001,99$.

Therefore $9.888 - f$ will be $= 9.888 - 0.001,99 = 9.886,01$; that is, the third near value of x in the propoſed equation $x^4 - 3xx + 75x = 10,000$ will be 9.886,01. Q. E. I.

Art. 25. If x is $= 9.886,01$, we ſhall have

$xx\ (= \overline{9.886,01}^2) = 97.733,193,720,1$,

and $x^3\ (= \overline{9.886,01}^3) = 966.191,330,448,845,801$,

and $x^4\ (= \overline{9.886,01}^4) = 9551.777,154,730,594,077,144,01$,

and conſequently $3xx\ (= 3 \times 97.733,193,720,1) = 293.199,591,160,3$

and $75x\ (= 75 \times 9.886,01) = 741.450,75$,

and

and $x^4 + 75x = 10{,}293.227{,}904{,}730{,}594{,}077{,}144{,}01$,
and $x^4 + 75x - 3xx$, or $x^4 - 3xx + 75x$, =
10,293.227,904,730,594,077,144,01
— 293.199,591,160,3
= 10,000.028,313,570,294,077,144,01; which is a little greater than 10,000, or the abſolute term of the propoſed equation $x^4 - 3xx + 75x = 10{,}000$. And conſequently 9.886,01 will be a little greater than the true value of x in that equation.

A third Proceſs of Mr. Raphſon's Method of Approximation.

Art. 26. In order to obtain a ſtill nearer value of x, let g be put for the exceſs of 9.886,01 above it's true value. And we ſhall then have $x = 9.886{,}01 - g$,
and $xx\ (= \overline{9.886{,}01 - g}^2 = \overline{9.886{,}01}^2$
$- 2 \times 9.886{,}01 \times g + \&c) =$
$97.733{,}193{,}720{,}1 - 19.772{,}02 \times g + \&c$
and $x^4\ (= \overline{9.886{,}01 - g}^4 = \overline{9.886{,}01}^4 - 4 \times$
$\overline{9.886{,}01}^3 \times g + \&c$
$= \overline{9.886{,}01}^4 - 4 \times 966.191{,}330{,}448{,}845{,}801 \times g + \&c$
$= \overline{9.886{,}01}^4 - 3864.765{,}321{,}795{,}383{,}204 \times g + \&c)$
$= 9551.777{,}154{,}730{,}594{,}077{,}144{,}01$
$- 3864.765{,}321{,}795{,}383{,}204 \times g + \&c,$
and $3xx\ (= 3 \times \overline{97.733{,}193{,}720{,}1 - 19.772{,}02 \times g + \&c}$
$= 3 \times 97.733{,}193{,}720{,}1 - 3 \times 19.772{,}02 \times g + \&c)$
$= 293.199{,}591{,}160{,}3 - 59.316{,}06 \times g + \&c,$
and $75x\ (= 75 \times \overline{9.886{,}01 - g} = 75 \times 9.886{,}01 - 75g)$
$= 741.450{,}75 - 75g$, and conſequently $x^4 - 3xx + 75x =$ the compound quantity

9551.

$$\left\{\begin{array}{lll} 9551.777.154,730,594,077,144,01 & -3864.765,321,795,383,204 \times g + \&c \\ -293.199,591,160,3 & + 59.316,06 \times g - \&c \\ +741.450,75 & - 75 \times g \end{array}\right\}$$

$$= 10,000.028,313,570,294,077,144,01 - 3880.449,261,795,383,204 \times g + \&c.$$

Therefore this laſt quantity 10,000.028,313,570,294,077,144,01 — 3880.449,261,795,383,204 × g + &c will be nearly = 10,000.

Therefore (adding 3880.449,261,795,383,204 × g to both ſides,) we ſhall have 10,000.028,313,570,294,077,144,01 = 10,000 + 3880.449,261,795,383,204 × g, and (ſubtracting 10,000 from both ſides,) 3880.449,261,795,383,204 × g = 0.028,313,570,294,077,144,01, and, laſtly, (dividing both ſides by 3880.449,261,795,383,204) g = 0.000,007,296,467.

Therefore x, or 9.886,01 — g, will be = 9.886,01 — 0.000,007,296,467 = 9.886,002,703,533; that is, the fourth near value of x in the propoſed equation $x^4 - 3xx + 75x = 10,000$, obtained by this third proceſs of Mr. Raphſon's method of approximation, will be 9.886,002,703,533. Q. E. I.

Art. 27. Of this number 9.886,002,703,533, the firſt nine figures, 9.886,002,70, are the ſame with the firſt nine figures of the laſt value of x found above in art. 20 by Dr. Halley's method of approximation, to wit, 9.886,002,700. And therefore we may be confident that theſe nine figures, 9.886,002,70 are exact, or the ſame with the firſt nine figures of the true value of x in the propoſed

poſed equation $x^4 - 3xx + 75x = 10{,}000$. And I believe that the firſt eleven figures, to wit, 9.886,002, 703,5, of the value obtained in the foregoing article, to wit, 9.886,002,703,533, are exact.

Dr. Halley makes the more accurate value of x in this equation, which he has obtained by the ſecond proceſs of his method of approximation, to be 9.886,260,393,649,5, (ſee above, page 15.) which, he tells us, ſcarce exceeds the truth by 2 in the laſt figure. But from the agreement of the two values of x found in the preceeding articles by the application of both his and Mr. Raphſon's methods of approximation, in the firſt nine figures 9.886,002,70, we may ſafely conclude that theſe nine figures are exact, and conſequently that the value of x aſſigned by Dr. Halley is erroneous in all the figures 260,393,649,5 that come after the four firſt figures 9.886.

I now proceed to conſider Dr. Halley's third and laſt example.

EXAMPLE III.—A BIQUADRATICK EQUATION.

Art. 28. Let it be required to reſolve the biquadratick equation $-x^4 + 80x^3 - 1998x^2 + 14{,}937x = 5000$, or $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, by Dr. Halley's method of approximation.

An Inveſtigation, by Conjectures and Trials, of a, *or the firſt near Value of* x.

In order to find a firſt near value of one of the roots of this biquadratick equation (for it has four real and affirmative roots,) to be made the baſis of a ſecond approximation to it's true value in the manner recommended by Dr. Halley, we may proceed as follows:

Let us ſubſtitute two or three very eaſy numbers, conſiſting of two figures each, inſtead of x, in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, which forms the left-hand ſide of the propoſed equation, in order to ſee whether either of the values reſulting from ſuch ſubſtitutions will be nearly equal to 5000, or the abſolute term of the ſaid equation. And, firſt, let us ſuppoſe x to be equal to 10.

Then will the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ be $= 14{,}937 \times 10 - 1998 \times 100 + 80 \times 1000 - 10{,}000$ $(= 149{,}370 - 199{,}800 + 80{,}000 - 10000 = 229{,}370 - 209{,}800) = 19{,}570$; which is conſequently greater than the abſolute term 5000. We will, therefore, in the ſecond place, ſuppoſe x to be $= 12$.

Then we ſhall have $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 14{,}937 \times 12 - 1998 \times 144 + 80 \times 1728 - 20{,}736$ $(= 179{,}244 - 287{,}712 + 138{,}240 - 20{,}736 = 317{,}484 - 308{,}448) = 9{,}036$; which is leſs than the former reſult, 19,570, but yet is greater than the abſolute term 5000.

We will therefore, in the third place, ſuppoſe x to be $= 13$.

And then we ſhall have $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 14{,}937 \times 13 - 1998 \times 169 + 80 \times 2197 - 28{,}561$ $(= 194{,}181 - 337{,}662 + 175{,}760 - 28{,}561 = 369{,}941 - 366{,}223) = 3{,}718$; which is leſs than the abſolute term 5000.

Therefore, if we ſuppoſe x to increaſe from 12 to 13, the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will have decreaſed at the ſame time from 9,036 to 3,718, and therefore muſt, at ſome inſtant of time during

during the ſaid decreaſe, have been equal to the intermediate quantity 5000; or, in other words, there will be a quantity greater than 12, but leſs than 13, that will be a root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Further, ſince the reſult of the ſubſtitution of 12 inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is 9,036, and the reſult of the ſubſtitution of 13 inſtead of x in the ſame quantity is 3,718, which differs leſs than the former reſult, or 9,036, from the abſolute term 5000, it ſeems reaſonable to ſuppoſe that the true value of x in the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ will approach nearer to 13 than to 12; and it alſo ſeems reaſonable to ſuppoſe that the difference of the two extreme numbers 12 and 13, correſponding to the reſults 9036 and 3718, will be to the difference of 12 and x in nearly the ſame proportion as the difference of the ſaid reſults 9036 and 3718 to the difference of the former reſult 9036, (which correſponds to 12) and the abſolute term 5000, or that $13 - 12$ will be to $x - 12$ in nearly the ſame proportion as $9036 - 3718$ is to $9036 - 5000$, or that 1 will be to $x - 12$ in nearly the ſame proportion as 5318 is to 4036, and conſequently that $x - 12$ will be nearly equal to $\frac{1 \times 4036}{5318}$, or to 0.7, or that x will be nearly equal to $12 + 0.7$, or 12.7. And therefore we will take 12.7 for our firſt near value of x, which we denote by the letter a, and will make it the baſis of our further approximation to the true value of x in the propoſed equation in the method recommended by Dr. Halley.

A more exact Determination of the Value of x *by Dr. Halley's Method of Approximation.*

Art. 29. As we do not yet know whether 12.7 is greater, or leſs, than x, let 12.7 be ſubſtituted inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the reſult will be greater, or leſs, than the abſolute term 5000, and from that circumſtance to determine whether 12.7 is greater or leſs than x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, if x is $= 12.7$, we ſhall have $xx\ (= \overline{12.7}^2) = 161.29$, and $x^3\ (= \overline{12.7}^3) = 2048.383$, and $x^4\ (= \overline{12.7}^4) = 26{,}014.4641$, and conſequently $14{,}937x\ (= 14{,}937 \times 12.7) = 189{,}699.9$, and $1998x^2\ (= 1998 \times 161.29) = 322{,}257.42$, and $80x^3\ (= 80 \times 2048.383) = 163{,}870.640$, and the whole quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4\ (= 189{,}699.9 - 322{,}257.42 + 163{,}870.640 - 26{,}014.4641 = 353{,}570.5400 - 348{,}271.8841) = 5298.6559$; which is ſomewhat greater than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore, while x increaſes from 12 to 12.7, the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will have decreaſed only from 9036 to 5298.6559. But we have ſeen that, when x is $= 13$, the ſaid compound quantity will be $= 3718$. Therefore, while x increaſes from 12.7 to 13, the ſaid compound quantity will decreaſe from 5298.6559 to 3718, and conſequently will, at ſome inſtant of time during it's ſaid decreaſe, be equal to the intermediate quantity 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$;

5000; and therefore there will be ſome value of x greater than 12.7, but leſs than 13, that, being ſubſtituted inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, will make the ſaid compound quantity be equal to 5000; or, in other words, there will be a root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ that will be greater than 12.7, but leſs than 13.

Art. 30. Having thus diſcovered that the true value of that root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ which we are now ſeeking, is greater than 12.7, let e be put for it's unknown exceſs above 12.7, ſo that x ſhall be $= 12.7 + e$, or $a + e$.

Then we ſhall have

$$xx\ (= \overline{a+e}|^2) = aa + 2ae + ee,$$

and $x^3\ (= \overline{a+e}|^3) = a^3 + 3a^2e + 3ae^2 + e^3$,

and $x^4\ (= \overline{a+e}|^4) = a^4 + 4a^3e + 6a^2e^2 + 4ae^3 + e^4$,

and conſequently

$14{,}937x\ (= 14937 \times \overline{a+e}) = 14{,}937a + 14{,}937e$,

and $1998x^2\ (= 1998 \times \overline{aa + 2ae + ee}) =$

$$1998aa + 3996ae + 1998ee,$$

and $80x^3\ (= 80 \times \overline{a^3 + 3a^2e + 3ae^2 + e^3}) =$

$$80a^3 + 240a^2e + 240ae^2 + 80e^3;$$

and conſequently the whole quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{rrrrrrrrr} & 14{,}937a & + & 14{,}937e & & & & & \\ - & 1998aa & - & 3996ae & - & 1998ee & & & \\ + & 80a^3 & + & 240a^2e & + & 240ae^2 & + & 80e^3 & \\ - & a^4 & - & 4a^3e & - & 6a^2e^2 & - & 4ae^3 & - e^4. \end{array}\right\}$$

But the quadrinomial quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is $= 5000$.

Therefore the laſt-mentioned compound quantity, involving the unknown quantity e inſtead of x, will alſo be $= 5000$.

Now let the three laſt terms of this laſt compound quantity, to wit, the terms $+ 80\,e^3 - 4\,ae^3 - e^4$, be expunged from it, on account of their ſmallneſs in compariſon of the terms in the ſecond and third vertical columns of the ſaid compound quantity, which involve in them the ſquare and the ſimple power of the unknown quantity e. And we ſhall then have the remaining compound quantity, to wit, the quantity

$$\left\{\begin{array}{rrrr} 14{,}937\,a & + & 14{,}937\,e & \\ - \ 1998\,a^2 & - & 3996\,ae & - \ 1998\,e^2 \\ + \ 80\,a^3 & + & 240\,a^2e & + \ 240\,ae^2 \\ - \ a^4 & - & 4\,a^3e & - \ 6\,a^2e^2 \end{array}\right\}$$

nearly $= 5000$.

Now the firſt vertical column of terms in this equation, to wit, $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$, has already been ſhewn (in art. 29,) to be $= 5298.6559$; and $3996 \times a$ is $(= 3996 \times 12.7) = 50{,}749.2$; and $240\,a^2$ is $(= 240 \times \overline{12.7}^2 = 240 \times 161.29) = 38{,}709.60$; and $4\,a^3$ is $(= 4 \times \overline{12.7}^3 = 4 \times 2048.383) = 8193.532$; and $240\,a$ is $(= 240 \times 12.7) = 3048.0$; and $6\,a^2$ is $(= 6 \times \overline{12.7}^2 = 6 \times 161.29) = 967.74$.

Therefore we ſhall have the compound quantity

$$\left\{\begin{array}{rrrrrr} 5298.6559 & + & 14,937 & e & & \\ & - & 50,749.2 & \times e - & 1998 & \times e^2 \\ & + & 38,709.60 & \times e + & 3048.0 & \times e^2 \\ & - & 8,193.532 & \times e - & 967.74 & \times e^2 \end{array}\right\},$$

nearly = 5000, or the compound quantity

$$\left\{\begin{array}{rrrrrr} 5298.6559 & + & 53,646.60 & \times e - & 2965.74 & \times e^2 \\ & - & 58,942.732 & \times e + & 3048.0 & \times e^2 \end{array}\right\}$$

= 5000, or the compound quantity

$5298.6559 - 5296.132 \times e + 82.26 \times e^2 = 5000$. Therefore (adding $5296.132 \times e$ to both ſides,) we ſhall have $5298.6559 + 82.26 \times e^2 = 5000 + 5296.132 \times e$; and (ſubtracting $82.26 \times e^2$ from both ſides,) we ſhall have $5298.6559 = 5000 + 5296.132 \times e - 82.26 \times e^2$; and, laſtly, (ſubtracting 5000 from both ſides,) we ſhall have $5296.132 \times e - 82.26 \times e^2 = 298.6559$; which is a quadratick equation properly prepared for reſolution, and which we will therefore now proceed to reſolve, in order to obtain the value of e, or of the exceſs of the true value of x in the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ above a, or 12.7, it's firſt near value.

Art. 31. Let all the terms of this equation $5296.132 \times e - 82.26 \times e^2 = 298.6559$ be divided by 82.26, the co-efficient of e^2. And we ſhall then have $64.382{,}834{,}9 \times e - e^2 = 3.630{,}633{,}357{,}646{,}48$. Now let both ſides of this equation be ſubtracted from the ſquare of half the co-efficient of e, that is, from the ſquare of half of 64.382,834,9, or the ſquare of 32.191,417,4, or from 1036.287,354,221,022,76. And we ſhall have $\overline{32.191{,}417{,}4}^2 - 64.382{,}834{,}9 \times e + e^2$ (= 1036.287,

354,221,022,76 − 3.630,633,357,646,48) = 1032.656, 720,863,376,28. Therefore the ſquare-root of the trinomial quantity $\overline{32.191,417,4}^2 - 64.382,834,9 \times e + e^2$ will be equal to the ſquare-root of 1032.656,720, 863,376,28, that is, to 32.134,976,5.

But the trinomial quantity $\overline{32.191,417,4}^2 - 64.382,834,9 \times e + e^2$ has two ſquare-roots, to wit, the binomial quantity 32.191,417,4 — e and the binomial quantity e — 32.191,417,4. We muſt therefore now inquire which of theſe two ſquare-roots will enable us to find the value of e required on the preſent occaſion, or that value of it which is equal to the exceſs of the root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ above 12.7. Now we know that this exceſs is leſs than 0.3, becauſe x is leſs than 13. Therefore the latter ſquare-root of the ſaid trinomial quantity, to wit, the binomial quantity e − 32.191,417,4, (in which e is greater than 32.191,417,4.) cannot be that which is ſuited to our preſent purpoſe; and conſequently we muſt make uſe of the other ſquare-root of the ſaid trinomial quantity, to wit, the binomial quantity 32.191, 417,4 — e. We ſhall therefore have 32.191,417,4 − e = 32.134,976,5; and conſequently (adding e to both ſides,) 32.191,417,4 will be = 32.134,976,5 + e, and e will be (= 32.191,417,4 − 32.134,976,5) = 0.056,440,9. Therefore $a + e$, or 12.7 + e, will be (= 12.7 + 0.056,440,9) = 12.756,440,9; that is, the ſecond near value of x, or the root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, obtained by this firſt proceſs of Dr. Halley's method of approximation, will be 12.756,440,9. Q. E. I.

This value of x is exact in the first seven figures 12.756,44, it's more accurate value, as computed by both Dr. Halley and Dr. Wallis, being 12.756,441,794,480, 744,02.

A Correction of the foregoing, or second, near Value of x, *suggested by Dr. Halley.*

Art. 32. Dr. Halley on this occasion points out a correction to be made to the value of x just now obtained, to wit, the number 12.756,440,9, without entering upon a compleat second process of his method of approximation, and tells us that we may, by this correction, find the value of x to be = 12.756,441,794,48, or to thirteen figures, all exact. This correction, if I understand it, may be explained as follows.

The compleat transformed equation obtained in art. 30 by substituting $a + e$ instead of x in the terms of the original equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, is

$$\left\{\begin{array}{rrrrr} 14{,}937a & + & 14{,}937e & & \\ - 1998a^2 & - & 3996ae & - & 1998e^2 \\ + 80a^3 & + & 240a^2e & + 240ae^2 & + 80e^3 \\ - a^4 & - & 4a^3e & - 6a^2e^2 - 4ae^3 & - e^4 \end{array}\right\}$$

$= 5000.$

And by substituting 12.7, or the value of a, instead of a in all the terms of this equation, except $80e^3 - 4ae^3 - e^4$, this equation will be converted into the following equation, to wit, $5298.6559 - 5296.132 \times e + 82.26 \times e^2 + 80e^3 - 4ae^3 - e^4 = 5000$.

Add $5296.132 \times e$ to both sides: and we shall have $5298.6559 + 82.26 \times e^2 + 80e^3 - 4ae^3 - e^4 = 5000 + 5296.132 \times e$.

Subtract

Subtract $82.26 \times e^2$ from both ſides: and we ſhall have $5298.6559 + 80e^3 - 4ae^3 - e^4 = 5000 + 5296.132 \times e - 82.26 \times e^2$.

Laſtly, ſubtract 5000 from both ſides: and we ſhall have $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 4ae^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 4 \times 12.7 \times e^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 80e^3 - 50.8 \times e^3 - e^4$, or $5296.132 \times e - 82.26 \times e^2 = 298.6559 + 29.2 \times e^3 - e^4$. Therefore (dividing all the terms by 82.26, (the co-efficient of e^2,) we ſhall have $64.382,834,913,688 \times e - e^2 = 3.630,633,357,646,486,749,331,388 + 0.354,972,039,873, \times e^3 - \frac{e^4}{82.26}$.

Now let the value of e that we have already found, to wit, 0.05644, be inſerted inſtead of e in the two terms $0.354,972,039,873 \times e^3$ and $\frac{e^4}{82.26}$; and for this purpoſe let ϵ, or 0.05644, be raiſed to it's third and fourth powers.

Then we ſhall have $\epsilon^3 = 0.003,185,473,6$, and $\epsilon^3 = 0.000,179,788,129,984$, and $\epsilon^4 = 0.000,010,147,242,056,296,96$. Therefore $0.354,972,039,873 \times e^3$ will be $(= 0.354,972,039,873 \times 0.000,179,788,129,984) = 0.000,063,819,759,245,372,554,852,032$; and $\frac{e^4}{82.26}$ will be $= \frac{0.000,010,147,242,056,296,96}{82.26} = 0,000,000,$

123,

123,355,726,432,007,780; and conſequently 0.354,972,039,873 × e^3 — $\frac{e^4}{82.26}$ will be (= 0.000,063,819,759,245,372,554,852,032 — 0.000,000,123,355,726,432,007,780) = 0.000,063,696,403,518,940,547,072,032; and 3.630,633,357,646,486,749,331,388 + 0.354,972,039,873 × e^3 — $\frac{e^4}{82.26}$ will be (= 3.630,633,357,646,486,749,331,388 + 0.000,063,696,403,518,940,547,072,032) = 3.630,697,054,050,005,689,878,460,032. Therefore 64.382,834,913,688 × e — e^2 will be = 3.630,697,054,050,005,689,878,460,032; which quadratick equation we muſt now reſolve.

Art. 33. The half of 64.382,834,913,688 (the co-efficient of e) is 32.191,417,456,844, the ſquare of which is 1036.287,357,880,800,614,602,440,336. From this number let both ſides of the equation 64.382,834,913,688 × e — e^2 = 3.630,697,054,050,005,689,878,460,032 be ſubtracted. And we ſhall have the trinomial quantity $\overline{32.191,417,456,844}^2$ — 64.382,834,913,688 × e + e^2 (= 1036.287,357,880,800,614,602,440,336 — 3.630,697,054,050,005,689,878,460,032) = 1032.656,660,826,750,608,912,561,875,968. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $\overline{32.191,417,456,844}$ — e = 32.134,975,662,457; and conſequently (adding e to both ſides,) we ſhall have 32.191,417,456,844 = 32.134,975,662,457 + e, and (ſubtracting 32.134,975,662,459 from both ſides) e = 0.056,441,794,387. Therefore a + e, or 12.7 + e will be (= 12.7 + 0.056,441,794,387) = 12.756,441,794,387; that is, the value of x in the propoſed equation 14,937 x —

1998 x^2

$1998x^2 + 80x^3 - x^4 = 5000$ will be 12.756,441, 794,387. Q. E. I.

A Remark on the foregoing Correction of the ſecond near Value of x, *ſuggeſted by Dr. Halley.*

Art. 34. This number 12.756,441,794,387 is exact in the firſt eleven figures 12.756,441,794, whereas the former number, obtained in art. 31, to wit, 12.756,440,9, was exact in only the firſt ſeven figures 12.756,44; ſo that this correction has given us four new figures of the value of x exact. And Dr. Halley makes the number obtained by means of the correction exact to two figures more, it being according to his calculation 12.756,441, 794,48, which conſiſts of thirteen figures, which are all exact. But even this additional number of exact figures ſeems to be hardly worth the trouble of going through the calculations that are neceſſary to obtain them. If more figures of the value of x are required than are afforded by the firſt proceſs of Dr. Halley's method of approximation, I ſhould rather be inclined to enter upon a compleat ſecond proceſs of it, grounded upon the value found by the firſt proceſs, than to have recourſe to this correction of the firſt proceſs; partly becauſe I believe the labour of calculation in performing a compleat ſecond proceſs will hardly be greater than that of applying the foregoing correction to the firſt proceſs, and partly, and chiefly, becauſe I think the reaſonings employed in a ſecond proceſs are clearer and more ſatisfactory than thoſe employed in making the correction of the firſt proceſs, and leſs likely to lead the calculator into a miſtake, ſuch as that of adding a quantity where it ſhould be ſubtracted, or the contrary, or of omitting ſome neceſſary diviſion of the

the new quantities taken into the calculation, or not carrying the diviſion far enough. But the comparative merits of a compleat ſecond proceſs of Dr. Halley's method of approximation to the root of an equation, and of the foregoing correction of the firſt proceſs of it, will be better underſtood by exhibiting a compleat ſecond proceſs of it in the caſe of the foregoing equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, grounded on the value of x obtained by the firſt proceſs, which was 12.756,440,9, or rather on 12.756,44, the firſt ſeven figures of that value, becauſe thoſe figures will be ſufficient to enable us to find the next value of x exact to about 21 places of figures; and therefore I ſhall now proceed to exhibit the ſaid ſecond proceſs in the ſame full manner as the former proceſs.

An Inveſtigation of a third near Value of x *by a ſecond Proceſs of Dr. Halley's Method of Approximation.*

Art. 35. We will therefore, in the firſt place, ſubſtitute the number 12.756,44 inſtead of x in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to diſcover whether the value of that quantity reſulting from ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, and from that circumſtance to determine whether the number 12.756,44 is greater, or leſs, than the true value of x in that equation.

Now, if x is $= 12.756{,}44$, we ſhall have

$x\,x\ (= \overline{12.756{,}44}^2) = 162.726{,}761{,}473{,}6$;

and $x^3\ (= \overline{12.756{,}44}^3) = 2075.814{,}169{,}132{,}289{,}984$;

and

and x^4 ($= \overline{12.756,44}^4$) $= 26,479.998,899,685,909,243,496,96$;

and consequently $14,937\,x$ ($= 14,937 \times 12.756,44$) $= 190,542.944,28$;

and $1998\,x\,x$ ($= 1998 \times 162\,726,761,173,6$) $= 325,128.069,424,252,8$;

and $80\,x^3$ ($= 80 \times 2075.814,169,132,289,984$) $= 166,065.133,530,583,198,720$.

Therefore the whole compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be (=

$$\left\{\begin{array}{l} 190,542.944,28 - 325,128.069,424,252,8 \\ +\ 166,065.133,530,583,198,720 - 26,479.998,899,685, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 909,243,496,96 \end{array}\right\}$$

$$= \left\{\begin{array}{l} 356,608.077,810,583,198,720 \\ -351,608.068,323,938,709,243,496,96 \end{array}\right\}$$

$= 5000.009,486,644,489,476,503,04$; which is a very little greater than 5000, or the abſolute term of the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. Therefore, for the reaſons given above in art. 29, we may conclude that the number 12.75644 is leſs than the true value of x in that equation.

Art. 36. Having thus diſcovered that the number 12.75644 is leſs than the true value of x in the equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, let us put c for the number 12.75644, and f for the unknown exceſs of x above it, ſo that x will be equal to $c + f$.

Then

Then we ſhall have $xx\ (= \overline{c+f}|^2) = cc + 2cf + ff$, and $x^3\ (= \overline{c+f}|^3) = c^3 + 3c^2f + 3cf^2 + f^3$, and $x^4\ (= \overline{c+f}|^4) = c^4 + 4c^3f + 6c^2f^2 + 4cf^3 + f^4$, and $14{,}937x\ (= 14{,}937 \times \overline{c+f}) = 14{,}937c + 14{,}937f$, and $1998x^2\ (= 1998 \times \overline{cc + 2cf + ff}) = 1998cc + 3996cf + 1998ff$, and $80x^3\ (= 80 \times \overline{c^3 + 3c^2f + 3cf^2 + f^3}) = 80c^3 + 240c^2f + 240cf^2 + 80f^3$. And conſequently the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{rrrrrrrrr} & 14{,}937c & + & 14{,}937f & & & & & \\ - & 1998cc & - & 3996cf & - & 1998ff & & & \\ + & 80c^3 & + & 240c^2f & + & 240cff & + & 80f^3 & \\ - & c^4 & - & 4c^3f & - & 6c^2ff & - & 4cf^3 & - f^4. \end{array}\right\}$$

But the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the other compound quantity, which involves in it's terms the unknown quantity f inſtead of x, will alſo be equal to 5000.

Now let the three laſt terms, $+ 80f^3 - 4cf^3 - f^4$, be expunged from the ſaid compound quantity, on account of their extream ſmallneſs in compariſon to the preceeding terms which involve the ſquare and the ſimple power of f. And the remaining compound quantity, to wit,

14,937

$$\left\{\begin{array}{rrrr} & 14{,}937\,c & + & 14{,}937\,f \\ - & 1998\,cc & - & 3996\,cf - 1998\,f^2 \\ + & 80\,c^3 & + & 240\,c^2 f + 240\,cf^2 \\ - & c^4 & - & 4\,c^3 f - 6\,c^2 f^2 \end{array}\right\}$$

will be, nearly, = 5000.

But it has been ſhewn in art. 35 that the firſt vertical column of terms in this equation, to wit, $14{,}937\,c - 1998\,cc + 80\,c^3 - c^4$ is = 5000.009,486,644,489,476,503,04; and $3996\,c$ is (= 3996 × 12.75644) = 50,974.734,24;

and $240\,c^2$ is (= 240 × 162.726,761,473,6)
= 39,054.422,753,664,0;

and $4\,c^3$ is (= 4 × 2075.814,169,132,289,984)
= 8303.256,676,529,159,936;

and $240\,c$ is (= 240 × 12.756,44) = 3061.545,60;

and $6\,c^2$ is (= 6 × 162.726,761,473,6)
= 976.360,568,841,6.

Therefore we ſhall have the compound quantity

$$\left\{\begin{array}{l} 5000.009,486,644,489,476,503,04 \\ \quad + 14{,}937\,f \\ \quad - 50{,}974.734,24 \times f \\ \quad + 39{,}054.422,753,664,0 \times f \\ \quad - 8{,}303.256,676,529,159,936 \times f \\ \qquad - 1998 \times f^2 \\ \qquad + 3061.545,60 \times f^2 \\ \qquad - 976.360,568,841,6 \times f^2 \end{array}\right\}$$

nearly = 5000, or the compound quantity

5000.

$$\left\{\begin{array}{l} 5000.009,486,644,489,476,503,04 \\ \quad + 53,991.422,753,664,0 \times f \\ \quad - 59,277.990,916,529,159,936 \times f \\ \quad - \quad 2974.360,568,841,6 \times f^2 \\ \quad + \quad 3061.545,60 \times f^2 \end{array}\right\}$$

nearly = 5000, or the compound quantity

$$\begin{array}{l} 5000.009,486,644,489,476,503,04 \\ \quad - 5286.568,162,865,159,936 \times f \\ \quad + \quad 87.184,431,158,4 \times f^2, \text{ nearly, } = 5000; \end{array}$$

and consequently (adding $5,286.568,162,865,159,936 \times f$ to both sides,) $5000.009,486,644,489,476,503,04 + 87.184,431,158,4 \times f^2 = 5000 + 5286.568,162,865,159,936 \times f$, and (subtracting $87.184,431,158,4 \times f^2$ from both sides,) $5000.009,486,644,489,476,503,04 = 5000 + 5286.568,162,865,159,936 \times f - 87.184,431,158,4 \times f^2$, and, lastly, (subtracting 5000 from both sides,) $5286.568,162,865,159,936 \times f - 87.184,431,158,4 \times f^2 = 0.009,486,644,489,476,503,04$; which is a quadratick equation properly prepared for resolution, and which we will therefore now proceed to resolve in order to obtain the value of f, or of the excess of the true value of x in the proposed equation $14,937x - 1998x + 80x^3 - x^4 = 5000$ above c, or 12.75644, it's second near value.

Art. 37. Let all the terms of this equation $5286.568,162,865,159,936 \times f - 87.184,431,158,4 \times f^2 = 0.009,486,644,489,476,503,04$ be divided by $87.184,431,158,4$, which is the co-efficient of f^2. And we shall then have $60.636,607,851,008,642,270,08 \times f - f^2$

and $1998x^2$ will be $(= 1998 \times \overline{aa + 2ae + ee})$
$= 1998aa + 3996ae + 1998ee,$
and $80x^3$ will be $(= 80 \times \overline{a^3 + 3a^2e + 3ae^2 + e^3})$
$= 80a^3 + 240a^2e + 240ae^2 + 80e^3,$
and conſequently the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{rrrrrrr} & 14{,}937a & + & 14{,}937e & & & \\ - & 1998a^2 & - & 3996ae & - & 1998e^2 & \\ + & 80a^3 & + & 240a^2e & + & 240ae^2 & + 80e^3 \\ - & a^4 & - & 4a^3e & - & 6a^2e^2 - 4ae^3 & - e^4. \end{array}\right\}$$

But the former compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is $= 5000.$

Therefore the latter compound quantity will alſo be $= 5000.$

Now let all the ſix terms of the ſaid latter compound quantity which involve in them the ſquare, cube, and fourth power, of the unknown quantity e, to wit, the terms $- 1998e^2 + 240ae^2 - 6a^2e^2 + 80e^3 - 4ae^3 - e^4$, be expunged from the ſaid compound quantity on account of their ſmallneſs in compariſon of the terms $+ 14{,}937e - 3996ae + 240a^2e - 4a^3e$, which involve only the ſimple power of e: and it is evident that the remaining compound quantity, to wit, the compound quantity $14{,}937a - 1998a^2 + 80a^3 - a^4 + 14{,}937e - 3996ae + 240a^2e - 4a^3e$, will be, nearly, $= 5000.$

But

But it has been ſhewn in art. 29 that, if a be equal to 12.7, (as it is here ſuppoſed to be,) the quadrinomial quantity $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$ will be $= 5298.6559$.

Therefore the compound quantity $5298.6559 + 14{,}937\,e - 3996\,ae + 240\,a^2e - 4\,a^3e$ will be nearly equal to 5000.

Further, ſince a is $= 12.7$, we ſhall have $a^2 = 161.29$, and $a^3 = 2048.383$, as is ſhewn in art. 29. Therefore $3996\,ae$ will be $(= 3996 \times 12.7 \times e) = 50{,}749.2 \times e$, and $240\,a^2e$ will be $(= 240 \times 161.29 \times e) = 38{,}709.60 \times e$, and $4\,a^3 \times e$ will be $(= 4 \times 2048.383 \times e) = 8193.532 \times e$.

Therefore the compound quantity $5298.6559 + 14{,}937\,e - 50{,}749.2 \times e + 38{,}709.60 \times e - 8193.532 \times e$ will be nearly $= 5000$, or the compound quantity $5298.6559 + 53{,}646.60 \times e - 58{,}942.732 \times e$ will be nearly $= 5000$, or the compound quantity $5298.6559 - 5296.132 \times e$ will be nearly $= 5000$. Therefore (adding $5296.132 \times e$ to both ſides,) we ſhall have $5298.6559 = 5000 + 5296.132 \times e$, and (ſubtracting 5000 from both ſides,) we ſhall have $5296.132 \times e = 298.6559$; which is a ſimple equation properly prepared for reſolution.

This ſimple equation $5296.132 \times e = 298.6559$ is reſolved by the ſingle operation of dividing both ſides of

the equation by the number 5296.132, which is the coefficient of the unknown quantity e. And by this division we ſhall find e to be $(= \frac{1298.6555}{5296.132}) = 0.056,39$. Therefore $a + e$, or $12.7 + e$, will be $(= 12.7 + 0.056,39) = 12.756,39$; that is, the ſecond near value of x in the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, obtained by this firſt proceſs of Mr. Raphſon's method of approximation, will be 12.756,39. Q. E. I.

Art. 39. Now let 12.75639 be ſubſtituted inſtead of x in the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, and from that circumſtance to determine whether the true value of x will be greater or leſs than 12.75639. This ſubſtitution will be as follows.

If x is $= 12.756,39$, we ſhall have

$x\,x\ (= \overline{12.75639}^2) = 162.725,485,832,1$,

and $x^3\ (= \overline{12.75639}^3) = 2075.789,760,213,742,419$,

and $x^4\ (= \overline{12.75639}^4) = 26,479.583,749,292,977,829,390,41$.

Therefore $14,937\,x$ will be $(= 14,937 \times 12.756,39) = 190,542,197,43$,

and $1998\,x^2$ will be $(= 1998 \times 162.725,485,832,1) = 325,125.520,692,535,8$,

and

and $80x^3$ will be $(= 80 \times 2075.789,760,243,742,119)$
$= 166,063.180,817,099,369,520,$
and consequently the whole compound quantity $14,937x$
$- 1998x^2 + 80x^3 - x^4$ will be $(= 190,542.197,442,$
$325,135,520,692,535,8 + 166,063.180,817,099,369,$
$520 - 26,479.583,749,292,977,829,390,41 = 356,$
$605.378,247,699,369,520 - 35.609.104,441,828,7775$
$829,396,41) = 5000.273,805,270,591,690,609,59;$
which is somewhat greater than 5000, or the absolute term of the proposed equation. Therefore, while x increases from 12.7 to 12.756,39, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will have decreased from 5298.6559 to 5000.273,805,270,591,690,609,595 and, while x increases further from 12.75639 to 13, the said compound quantity will have decreased farther from 5000.273,805,270,591,690,609,59, to 3718. Therefore there will be some value of x greater than 12.75639, but less than 13, that will make the said compound quantity be equal to the intermediate quantity 5000; or, in other words, the true value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be greater than 12.756,39, but less than 13.

Q. E. I.

A second Process of Mr. Raphson's Method of Approximation.

Art. 40. Now let c be put $=$ 12.756,39, and f for the unknown quantity by which the true value of x in the proposed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ exceeds c, or 12.756,39; so that x shall be $= c + f$, or $12.756,39 + f$.

Then we ſhall have $xx\,(=\overline{c+f}^{2}) = cc + 2cf + ff$, and $x^{3}(=\overline{c+f}^{3}) = c^{3} + 3c^{2}f + 3cf^{2} + f^{3}$, and $x^{4}(=\overline{c+f}^{4}) = c^{4} + 4c^{3}f + 6c^{2}ff + 4cf^{3} + f^{4}$, and conſequently

$14{,}937x\,(=14{,}937\times\overline{c+f}) = 14{,}937\times c + 14{,}937\times f$,

and $1998xx\,(=1998\times\overline{cc+2cf+ff})$ $1998cc + 3996cf + 1998ff$,

and $80x^{3}\,(=80\times\overline{c^{3}+3c^{2}f+3cf^{2}+f^{3}}) = 80c^{3} + 240c^{2}f + 240cf^{2} + 80f^{3}$,

and the whole compound quantity $14{,}937x - 1998xx + 80x^{3} - x^{4}$ = the compound quantity

$$\left\{\begin{array}{rrrrrrrrr} & 14{,}937c & + & 14{,}937f & & & & & \\ - & 1998cc & - & 3996cf & - & 1998ff & & & \\ + & 80c^{3} & + & 240c^{2}f & + & 240cff & + & 80f^{3} & \\ - & & c^{4} & - & 4c^{3}f & - & 6c^{2}ff & - 4cf^{3} & - f^{4}. \end{array}\right\}$$

But the compound quantity $14{,}937x - 1998x^{2} + 80x^{3} - x^{4}$ is $= 5000$.

Therefore the compound quantity

$$\left\{\begin{array}{rrrrrrrrr} & 14{,}937c & + & 14{,}937f & & & & & \\ - & 1998cc & - & 3996cf & - & 1998ff & & & \\ + & 80c^{3} & + & 240c^{2}f & + & 240cff & + & 80f^{3} & \\ - & & c^{4} & - & 4c^{3}f & - & 6c^{2}ff & - 4cf^{3} & - f^{4} \end{array}\right\}$$

will alſo be $= 5000$.

Now

Now let all the terms in this equation that involve either ff, f^3, or f^4, be expunged from it. And we ſhall then have the compound quantity

$$\left\{\begin{array}{rrr} & 14,937c & + 14,937f \\ - & 1998cc & - 3996cf \\ + & 80c^3 & + 240c^2f \\ - & c^4 & - 4c^3f \end{array}\right\} \text{ nearly} = 5000.$$

But it has been ſhewn in the laſt article that $14,937c - 1998cc + 80c^3 - c^4$, or $14,937 \times 12.756,39 - 1998 \times \overline{12.756,39}^2 + 80 \times \overline{12.756,39}^3 - \overline{12.756,39}^4$, is $= 5000.273,805,270,591,690,609,59$; and $3996c$ is $(= 3996 \times 12.756,39) = 50,974.534,44$; and

$240c^2$ is $(= 240 \times 162.725,485,832,1)$
$= 39,054.116,599,704,0$;

and $4c^3$ is $(= 4 \times 2075.789,760,213,742,119)$
$= 8303.159,040,854,968,476.$

Therefore the compound quantity

$$\left\{\begin{array}{l} 5000.273,805,270,591,690,609,59 \\ \quad + 14,937f \\ \quad - 50,974.534,44 \times f \\ \quad + 39,054.116,599,704,0 \times f \\ \quad - 8303.159,040,854,968,476 \times f \end{array}\right\}$$

will be, nearly, $= 5000$, or the compound quantity $5000.273,805,270,591,690,609,59 + 53,991.116,599,704,0 \times f - 59,277.693,480,854,968,476 \times f$ will be nearly $= 5000$, or the compound quantity 5000.273,

805,

805,270,591,690,609,59 — 5,286.576,881,150,968,476 × f will be, nearly, = 5000.

Therefore (adding 5,286.576,881,150,968,476 × f to both ſides,) we ſhall have 5000.273,805,270,591,690,609,59 = 5000 + 5,286.576,881,150,968,476 × f, and (ſubtracting 5000 from both ſides,) 5,286.576,881,150,968,476 × f = 0.273,805,270,591,690,609,59; which is a ſimple equation properly prepared for reſolution.

This equation is to be reſolved by the ſingle operation of dividing both ſides of it by 5,286.576,881,150,968,476, the co-efficient of the unknown quantity f; by which we ſhall find f to be $(= \frac{0.273,805,270,591,690,609,59}{5,286.576,881,150,968,476}) =$ 0.000,051,792. Therefore $x + f$, or 12.756,39 + f, will be (= 12.756,39 + 0.000,051,792) = 12.756,441,792; that is, the 3d near value of x in the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, which is obtained by this ſecond proceſs of Mr. Raphſon's method of approximation, will be 12.756,441,792. Q. E. I.

Of this number 12.756,441,792, the firſt ten figures, 12.756,441,79, are exact; it's more accurate value being 12.756,441,794,483,744,02, as has been ſeen in art. 37. But this degree of exactneſs may be attained by carrying this approximation by Mr. Raphſon's method one ſtep further; which may be done as follows.

Art. 41. Let the number (12,756,441,792,) obtained by the foregoing proceſs, be ſubſtituted inſtead of x in the compound quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, and, from that circumſtance, to determine whether the true value of x in that equation will be greater, or leſs, than the ſaid number.

Now, if x is $= 12{,}756{,}441{,}792$, we ſhall have

$xx\,(= \overline{12.756{,}441{,}792}^2) = 162{,}726{,}807{,}192{,}684{,}171{,}264$,

and $x^3\,(= \overline{12.756{,}441{,}792}^3) = 2075.815{,}043{,}951{,}482{,}558{,}968{,}975{,}065{,}088$,

and $x^4\,(= \overline{12.756{,}441{,}792}^4) = 26{,}480.013{,}779{,}125{,}008{,}935{,}590{,}937{,}951{,}694{,}483{,}357{,}696$.

Therefore $14{,}937\,x$ will be $(= 14{,}937 \times 12.756{,}441{,}792) = 190{,}542.971{,}047{,}104$,

and $1998\,x^2$ will be $(= 1998 \times 162.726{,}807{,}192{,}684{,}171{,}264) = 325{,}128.160{,}770{,}982{,}974{,}185{,}472$,

and $80\,x^3$ will be $(= 80 \times 2075.815{,}043{,}951{,}482{,}558{,}968{,}975{,}065{,}088) = 166{,}065.203{,}516{,}118{,}604{,}717{,}518{,}005{,}207{,}040$;

and conſequently $14{,}937\,x + 80\,x^3$ will be $(= 190{,}542.971{,}047{,}104 + 166{,}065.203{,}516{,}118{,}604{,}717{,}518{,}005{,}207{,}040) = 356{,}608.174{,}563{,}222{,}604{,}717{,}518{,}005{,}207{,}040$, and $1998\,x^2 + x^4$ will be $(= 325{,}128.160{,}770{,}982{,}974{,}185{,}472 + 26{,}480.013{,}779{,}125{,}008{,}935{,}590{,}937{,}$

937,951,694,483,357,696) = 351,608.174,550,107,983, 121,062,937,951,694,483,357,696; and $14,937x + 80x^2 - 1988x^3 - x^4$ will be (=

$$\left\{\begin{array}{r} 356,608.174,563,222,604,717,518,005,207,040, \\ -\ 351,608.174,550,107,983,121,062,937,951,694, \\ 483,357,696 \end{array}\right\}$$

= 5000.000,013,114,621,596,455,067,255,345,516,642, 304; that is, the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be = 5000.000,013,114,621,596, 455,067,255,345,516,642,304; which is a little greater than 5000, or the abſolute term of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. And therefore, for the reaſons given in art. 29, we may conclude that the true value of x in that equation will be greater than 12.756,441,792.

A third Proceſs of Mr. Raphſon's Method of Approximation.

Art. 42. Having now determined that the number 12.756,441,792, which was the near value of x obtained by the ſecond proceſs of Mr. Raphſon's method of approximation, is leſs than it's true value, let the ſaid near value 12.756,441,792 be denoted by the letter d, and the ſmall quantity by which the true value of x exceeds it be denoted by the letter g.

Then we ſhall have $x = d + g$, and conſequently $xx\ (= \overline{d+g}^2) = dd + 2dg + gg$, and $x^3\ (= \overline{d+g}^3) = d^3 + 3d^2g + 3dg^2 + g^3$, and $x^4\ (= \overline{d+g}^4) = d^4 + 4d^3g + 6d^2g^2 + 4dg^3 + g^4$.

Therefore

Therefore $14{,}937x$ will be $(= 14{,}937 \times \overline{d+g}) = 14{,}937d + 14{,}937 \times g$; and $1998x^2$ will be $(= 1998 \times \overline{dd + 2dg + gg}) = 1998dd + 3996dg + 1998gg$; and $80x^3$ will be $(= 80 \times \overline{d^3 + 3d^2g + 3dg^2 + g^3}) = 80d^3 + 240d^2g + 240dg^2 + 80g^3$. And consequently the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{l} 14{,}937d + 14{,}937g \\ -1998d^2 - 3996dg - 1998g^2 \\ +\ 80d^3 + 240d^2g + 240dg^2 + 80g^3 \\ -\ d^4 - 4d^3g - 6d^2g^2 - 4dg^3 - g^4. \end{array}\right\}$$

But the former compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is equal to 5000.

Therefore the latter compound quantity

$$\left\{\begin{array}{l} 14{,}937d + 14{,}937g \\ -1998d^2 - 3996dg - 1998g^2 \\ +\ 80d^3 + 240d^2g + 240dg^2 + 80g^3 \\ -\ d^4 - 4d^3g - 6d^2g^2 - 4dg^3 - g^4 \end{array}\right\}$$

will alſo be equal to 5000.

Now let the ſix terms $-1998g^2 + 240dg^2 - 6d^2g^2 + 80g^3 - 4dg^3 - g^4$ (which involve in them the ſquare, cube, and fourth power of the unknown quantity g,) be expunged out of this equation, on account of their ſmallneſs in compariſon of the four terms

+

$+ 14,937g - 3996dg + 240d^2g - 4d^3g$, which involve the ſimple power of g. And we ſhall then have the remaining compound quantity

$$\left\{\begin{array}{rrr} & 14,937d & + 14,937g \\ - & 1998d^2 & - 3996dg \\ + & 80d^3 & + 240d^2g \\ - & d^4 & - 4d^3g \end{array}\right\} \text{ nearly } = 5000.$$

But we have ſeen, in the laſt article, that, if d is equal to 12.756,441,792, the quadrinomial quantity $14,937d - 1998d^2 + 80d^3 - d^4$ will be equal to 5000.000, 013,114,621,596,455,067,255,345,516,642,304.

Therefore the compound quantity 5000.000,013,114, 621,596,455,067,255,345,516,642,304 $+ 14,937g -$ $3996dg + 240d^2g - 4d^3g$ will be nearly $= 5000$.

But, ſince d is $= 12.756,441,792$, we ſhall have d^2 $(= \overline{12.756,441,792}^2) = 162.726,807,192,684,171,$ 264, and d^3 $(= \overline{12.756,441,792}^3) = 2075.815,043,$ 951,482,558,968,975,065,088, and $3996d$ $(= 3996 \times$ $12.756,441,792) = 50,974.741,400,832$, and $240d^2$ $(= 240 \times 162.726,807,192,684,171,264) = 39,054.$ 433,726,244,201,103,360, and $4d^3$ $(= 4 \times 2075.815,$ 043,951,482,558,968,975,065,088) $= 8303.260,175,$ 805,930,235,875,900,260,352.

Therefore the compound quantity 5000.000,013,114, 621,596,455,067,255,345,516,642,304 $+ 14,937g -$ $3996dg + 240d^2g - 4d^3g$ will be $= 5000.000,$ 013,

013,114,621,596,455,067,255,345,516,642,304 +
14,937 g = 50,974.741,400,832 × g + 39,954.433,
726,244,201,103,360 × g = 8303.260,175,805,930,
235,875,900,260,352 × g = 5000.000,013,114,621,
596,455,067,255,345,516,642,304

+ 53,991.433,726,244,201,103,360 × g

= 50,278.001,876,627,930,235,875,900,260,352 × g

= 5000.000,013,114,621,596,455,067,255,345,516,642,304

= 5286.567,850,393,729,132,515,900,260,352 × g.

Therefore this laſt quantity 5000.000,013,114,621,596,455,067,255,345,516,642,304 = 5286.567,850,393,729,132,515,900,260,352 × g will be nearly = 5000; and conſequently (adding 5286.567,850,393,729,132,515,900,260,352 × g to both ſides,) 5000.000,013,114,621,596,455,067,255,345,516,642,304 will be = 5000 + 5286.567,850,393,729,132,515,900,260,352 × g, and, laſtly, (ſubtracting 5000 from both ſides,) 5,286.567,850,393,729,132,515,900,260,352 × g will be = 0.000,013,114,621,596,455,067,255,345,516,642,304; which is a ſimple equation properly prepared for reſolution.

Therefore g will be =

$$\frac{0.000,013,114,621,596,455,067,255,345,516,642,304}{5286.567,850,393,729,132,515,900,260,352} =$$

0.000,000,002,480,744,022,88; and conſequently $d + g$, or 12.756,441,792 + g, will be (= 12.756,441,792 + 0.000,000,002,480,744,022,88) = 12.756,441,794,480,

744,

744,022,88; that is, the fourth near value of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, obtained by this third proceſs of Mr. Raphſon's method of approximation, will be = 12.756,441,794,480,744,022,88. Q. E. I.

Art. 43. This number, 12.756,441,794,480,744,022,88, agrees with the number found by Dr. Wallis and Dr. Halley, for the value of x, to wit, the number 12.756,441,794,480,744,02, in all it's nineteen figures; and it agrees with the number found above in art. 37 by Dr. Halley's method of approximation, to wit, the number 12.756,441,794,480,744,022,60, in the firſt twenty figures 12.756,441,794,480,744,022: and therefore thoſe twenty figures are probably exact, and the true value of x is greater than 12.756,441,794,480,744,022, but leſs than 12.756,441,794,480,744,023. But this cannot be proved with certainty without ſubſtituting thoſe numbers inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$; which would be a work of great labour, and is not neceſſary to the object of this diſcourſe, which is to make a compariſon between Dr. Halley's and Mr. Raphſon's methods of reſolving high Algebräick equations by approximation, and to enable the reader to form a judgement concerning their reſpective merits, and determine for himſelf to which of the two he will give the preference: and this object, I hope, is now attained.

A SCHO-

A SCHOLIUM,

Concerning another Method of approximating to the Roots of High Equations, that may be called The Differential Method.

Art. 44. Having now gone through the reſolution of the biquadratick equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ by both Dr. Halley's and Mr. Raphſon's methods of approximation, and obtained the value of one of it's roots to a great degree of exactneſs, I will here ſtate to the reader another method that might have been taken to inveſtigate the ſame root to the ſame degree of exactneſs, that is totally different from both the foregoing methods. This method (which, I think, may be properly diſtinguiſhed by the appellation of the *Differential* method of approximating to the roots of high equations,) occurred to me as a natural and convenient artifice for obtaining the number 12.7, conſiſting of three figures, as the value of a, or the firſt near value of x in the aforeſaid equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$, which we were afterwards to make uſe of as the baſis, or ground-work, of the further approaches which were intended to be made to the true value of x in that equation by Dr. Halley's and Mr. Raphſon's methods of approximation. For the ſteps we took in art. 27 to find the ſaid number a, or 12.7, for a firſt near value of x, and the baſis of the ſubſequent approximations, were as follows:

Art. 45. In the firſt place we made a random gueſs that the number 10 (which is a number very eaſily managed in calculation,) might not be very different from the value of x in the propoſed equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$; and then, in order to try the juſtneſs of this conjecture, we ſubſtituted 10 inſtead of x in the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$; and we found that the value of the ſaid quantity reſulting from this ſubſtitution was 19,570; which is conſiderably greater than 5000, or the abſolute term of the propoſed equation. We therefore concluded that the true value of x in that equation muſt be greater, or leſs, than 10 by more than a ſingle unit, and we conjectured, in the ſecond place, that it might be nearly equal to 12.

We then tried this ſecond conjecture by ſubſtituting 12 inſtead of x in the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$, and we found the value of the ſaid quantity reſulting from this ſubſtitution to be 9,036; which is leſs than half the former reſult 19,570, but yet is conſiderably greater than the abſolute term 5000.

We therefore made a third conjecture that x might be nearly equal to 13, and we ſubſtituted 13 inſtead of x in the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$. And the value of the ſaid quantity reſulting from this ſubſtitution was found to be 3,718; which is leſs than the abſolute term 5000.

We then obſerved that, ſince, when x was equal to 12, the compound quantity $14{,}937x - 1998xx +$

 $80x^3$

$80x^3 - x^4$ was equal to 9,036, and, when x was equal to 13, the ſame compound quantity was equal to 3,718, it followed that, while x increaſed from 12 to 13, the ſaid compound quantity muſt have decreaſed from 9,036 to 3,718, and conſequently muſt, at ſome point of time during the ſaid decreaſe, have been equal to the intermediate magnitude 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. And therefore we concluded that there muſt be ſome value of x greater than 12, but leſs than 13, which would make the ſaid compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ be exactly equal to the intermediate magnitude 5000; or, in other words, that the true value of x in the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ muſt be greater than 12, but leſs than 13.

Art. 46. Having thus found that x muſt be greater than 12, but leſs than 13, we might have taken either of thoſe numbers for the baſis of a further approximation to the true value of x by either of the two methods of Dr. Halley and Mr. Raphſon. But, as I was deſirous of approaching a little nearer to the true value of x before I began the application of either of thoſe methods, I made uſe of the following conjectural and probable ſuppoſition, (which is ſimilar to that by which the logarithm of a number that is of an intermediate magnitude between two numbers that are very nearly equal to each other, and of which the logarithms are known, is derived from the logarithms of the ſaid two extreme numbers,) to wit, that, ſince the number 12, the number x, (or the

think that, if x and the two numbers that were nearly equal to it were to be taken much nearer to an equality with each other than the three numbers 12, x, and 13, (of which the greateſt exceeded the leaſt by 1, or the 12th part of the leaſt,) the aforeſaid ſuppoſed proportionality of the differences of the three contiguous numbers to the differences of the reſults of their ſubſtitution in the compound quantity $14,937x - 1998xx + 80x^3 - x^4$ would approach ſtill nearer to the truth, and conſequently that the number of figures that would be exact in the value of x obtained by means of ſuch ſuppoſed proportionality would be greater than it was in the former caſe. I therefore reſolved to try the effect of a ſecond conjecture founded on this ſuppoſed proportionality between the differences of x and two numbers very nearly equal to it and the differences of the three correſponding reſults of the ſubſtitution of x and the ſaid contiguous numbers in the ſaid compound quantity $14,937x - 1998xx + 80x^3 - x^4$. And with this view I took the near value of x obtained by the former proportion, to wit, the number 12.758, for the greater of the two contiguous numbers to x, and the number 12,756 for the leſſer of the ſaid two contiguous numbers, and I ſubſtituted, 1ſt, the number 12.758, and, 2ndly, the number 12.756, inſtead of x in the ſaid compound quantity $14,937x - 1998xx + 80x^3 - x^4$. The value of this compound quantity reſulting from the ſubſtitution of 12.758 was 4,991.762,652,593,904; and the value of it reſulting from the ſubſtitution of 12.756 was 5002.335,593,512,704. Therefore, while x increaſes from 12.756 to 12.758, the compound quantity $14,937x$ —

$- 1998 x x + 80 x^3 - x^4$ will decreaſe from 5002.335,593,512,704 to 4991.762,652,593,904, and conſequently will, at ſome one inſtant of time during the ſaid decreaſe, be equal to the intermediate magnitude 5000; or, in other words, the true value of x in the propoſed equation $14,937 x - 1998 x^2 + 80 x^3 - x^4 = 5000$ will be greater than 12.756, but leſs than 12.758.

Here therefore we have three contiguous numbers, to wit, 12.756, x, and 12.758, the greateſt of which exceeds the leaſt by only 0.002, or the 6378th part of the leaſt number 12.756; and we have the three correſponding quantities 5002.335,593,512,704, 5000, and 4991.762,652,593,904, which reſult from the ſubſtitution of the ſaid three contiguous numbers in the ſaid compound quantity $14,937 x - 1998 x x + 80 x^3 - x^4$. We may therefore ſuppoſe that the difference of the firſt and third of the ſaid three numbers 12.756, x, and 12.758, will be to the difference of the firſt and ſecond of the ſaid three numbers, to wit, 12.756 and x, in nearly the ſame proportion as the difference of the firſt and third of the ſaid reſults (correſponding to the ſaid three numbers,) is to the difference of the firſt and ſecond of the ſaid reſults, or that 12.758 — 12.756 will be to x — 12.756 in nearly the ſame proportion as 5002.335,593,512,704 — 4991.762,652,593,904 is to 5002.335,593,512,704 — 5000, or that 0.002 will be to x — 12.756 in nearly the ſame proportion as 10,572,940,918,800 is to 2.335,593,512,704; whence it will follow that x — 12.756 will be nearly

$$\left(= \frac{0.002 \times 2.335,593,512,704}{10.572,940,918,800} = \frac{0.004,671,187,025,408}{10.572,940,918,800}\right)$$

= 0.000,441,805, and therefore that x will be (= 0.000,441,805 + 12.756) = 12.756,441,805; of which value of x the firſt eight figures 12.756,441 are exact, and the ninth figure 8 is greater than the correſponding figure 7 in the true value of x, (which is 12.756,441, 794,480,744,022) by only an unit in the ſaid ninth figure, or 0.000,000,1.

Art. 49. The ſubſtitutions of 12.758 and 12.756 inſtead of x in the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ will be as follows:

If x is = 12.758, we ſhall have xx (= $\overline{12.758}|^2$) = 162.766,564, and x^3 (= $\overline{12.758}|^3$) = 2076.575, 823,512, and x^4 (= $\overline{12.758}|^4$) = 26,492.954,356, 366,096, and conſequently $14{,}937x$ (= 14,937 × 12.758) = 190,566.246, and $1998xx$ (= 1998 × 162.766,564) = 325,207.594,872, and $80x^3$ (= 80 × 2076.575, 823,512) = 166,026.065,880,960. Therefore the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will, upon this ſuppoſition, be

(= 190,566.246 − 325,207.594,872
+ 166,126.065,880,960 − 26,492.954,356,366,096
= 356,692.311,880,960 − 351,700.549,228,366,096)

= 4991.762,652,593,904; which is ſomewhat leſs than 5000, or the abſolute term of the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$.

And,

And, if x is = 12.756, we ſhall have xx (= $\overline{12.756}^2$) = 162.715,536, and x^3 (= $\overline{12.756}^3$) = 2075.599,377,216, and x^4 (= $\overline{12.756}^4$) = 26,476.345,655,767,296, and conſequently $14{,}937x$ (= 14,937 × 12.756) = 190,536.372, and $1998x^2$ (= 1998 × 162.715,536) = 325,105.640,928, and $80x^3$ (= 80 × 2075.599,377,216) = 166,047.950,177,280. Therefore the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will, upon this ſuppoſition, be

(= 190,536 372	−	325,105.640,928
+ 166,047.950,177,280	−	26,476.345,655,767,296
= 356,584.322,177,280	−	351,581.986,583,767,296)

= 5002.335,593,512,704; which is a little greater than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Art. 50. Since the former near value of x, to wit, 12.758, (which was obtained by means of the ſubſtitutions of the ſmall numbers 12 and 13 inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and of the proportion derived from thoſe ſubſtitutions,) was exact in the four firſt figures 12.75, and the laſt near value of x, to wit, 12.756,441,805, (which has been obtained by means of the ſubſtitutions of the numbers 12.758 and 12.756 inſtead of x in the ſaid compound quantity, and of the proportion derived from thoſe ſubſtitutions,) is exact in the eight firſt figures 12.756,441, and but a little too great in the ninth figure 8, we may conclude that every new proceſs of this differential method of reſolving equations will double the number

number of figures in the value of x, or the root ſought, that are already known; and therefore, in point of exactneſs, this method will be nearly upon a level with Mr. Raphſon's method of approximation, by which the figures of the root that are already known are doubled by every new proceſs. Nor is there much difference in the labour of performing the neceſſary calculations in this differential method of approximating to the roots of equations and in that of Mr. Raphſon, which conſiſts in ſubſtituting the binomial quantity $a + e$, or $a - e$, inſtead of x in the propoſed equation, and then reſolving the tranſformed equation, ariſing from ſuch ſubſtitution, as if it were a mere ſimple equation, or neglecting all the terms of it that involve in them any other power of the new unknown quantity e than it's ſimple power, or e itſelf. For in both methods it is neceſſary to ſubſtitute the former near value of x, to wit, the number 12.758, inſtead of x in the terms of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$; and in this *differential* method we are alſo obliged to ſubſtitute another near value of x, differing from the former near value 12.758 in the fifth, or laſt, figure, to wit, the number 12.756, inſtead of x in the terms of the ſaid compound quantity; the labour of making which ſecond ſubſtitution ſeems to be about equal to, or, perhaps, a little greater than, that of computing the compound co-efficient of e in the tranſformed equation

$$\left\{\begin{array}{rrrrr} & 14{,}937a & - & 14{,}937e & \\ - & 1998a^2 & + & 3996ae & - \&c \\ + & 80a^3 & - & 240a^2e & + \&c \\ - & a^4 & + & 4a^3e & - \&c \end{array}\right\} = 5000,$$

which

which would have ariſen, in Mr. Raphſon's method of proceeding, from the ſubſtitution of $12.758 - e$, or $a - e$, inſtead of x in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, to wit, the co-efficient $- 14{,}937 + 3996a - 240a^2 + 4a^3$, or $- 14{,}937 + 3996 \times 12.758 - 240 \times \overline{12.758}^2 + 4 \times \overline{12.758}^3$, or $- 14{,}937 + 3996 \times 12.758 - 240 \times 162.766{,}564 + 4 \times 2076.575{,}823{,}512$. But Mr. Raphſon's method ſeems to be more perſpicuous and ſcientifick than the other, as well as, in ſome degree, leſs laborious: and therefore, upon the whole, I think it preferable to this differential method of approximation, as well as to that of Dr. Halley, excepting in the beginning of the reſolution of an equation, or when we are endeavouring to find the quantity a, or the firſt near value of x, which is afterwards to be made the baſis, or ground-work, of a further approach to the true value of x by either Dr. Halley's or Mr. Raphſon's method of approximation: for to this purpoſe I think the ſaid differential method of approximation ſeems to be remarkably well adapted.

Art. 51. And in this uſe of the ſaid differential method for the diſcovery of a, or the firſt near value of x, which is to be made the baſis of a further approach to it's true value by Dr. Halley's or Mr. Raphſon's methods of approximation, I ſhould think it would be adviſeable to uſe as many figures of the value of x, obtained by it, as we have reaſon to think are exact, or nearly exact. Thus, for example, in reſolving the foregoing equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, I think it would have

have been better to make a equal to the number 12.758, conſiſting of five figures, obtained by the firſt eaſy proportion of 1 to $x - 12$ as 5318 to 4036, than to make it equal to the number 12.7, conſiſting of only three figures, which was obtained from the ſame proportion by taking only the firſt figure 0.7 of the quotient of the diviſion of 4036 by 5318. For, if we had taken a equal to all the ſaid five figures 12.758, we might, by the application of only one proceſs of Mr. Raphſon's method of approximation, have obtained a ſecond near value of x that would have been exact to at leaſt eight figures. And by this union of this differential method of approximating to the roots of high equations with Mr. Raphſon's method of reſolving them, by employing the former method in the firſt ſtage of the reſolution to obtain to as great a degree of exactneſs as one proceſs of the ſaid differential method will enable us to find it, the value of a, or the firſt near value of the root ſought, and then proceeding to make uſe of a, or the near value of the root x ſo obtained, as a baſis for a further approximation to the true value of the ſaid root by Mr. Raphſon's method, the calculator will, as I conceive, obtain the value of the root ſought to any propoſed degree of exactneſs with the leaſt trouble and the leaſt perplexity poſſible.

Art. 52. I will obſerve, however, that the learned Dr. Hutton, of Woolwich Academy (in his late compendious collection of tracts on the different branches of the Mathematicks in two volumes, octavo, drawn-up for the uſe of the Cadets in the Military Academy at Woolwich,) recommends this differential method of reſolving high equations

equations by approximation (which he calls *The Method of Trial and Error*, or *The Double Rule of False Position*,) for the compleat resolution of such equations, independently of, and in preference to, all other methods of resolving them whatsoever.'

End of the Scholium begun in Art. 44, *page* 97.

Art. 53. The foregoing biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ has three other roots, or, in the language of modern Algebraists, three other real and affirmative roots, of which the least is less than 1, and is equal to 0.350,987,04, &c, and the other two are greater than 30, being equal to 32.060,290, &c, and 34.832,280, &c, as I have found upon an investigation of them by Mr. Raphson's method of approximation, if no mistakes have been made in the calculations that were necessary for that purpose. The investigation of the least of these three roots, to wit, 0.350,987,04 &c, is given in the 3d volume of the *Scriptores Logarithmici*, pages 722, 723, 724.

Mr. Raphson's Observations on Monsieur de Lagny's and Dr. Halley's claims to the merit of having invented their Methods of resolving Equations by Approximation.

Art. 54. Before I conclude this discourse I will insert two passages from Mr. Raphson's appendix to the second edition of his ingenious and useful treatise on the resolution of equations of all orders by approximation, intitled *Analysis Æquationum Universalis*, in which he puts-in his claim to the merit of having invented this method of resolving

reſolving equations, and declares that he had had the ſame thought which Monſieur de Lagny and Dr. Halley afterwards purſued, of making uſe of the terms that involve the ſquare and other higher powers of the unknown quantity contained in the tranſformed equation, in order to obtain the value of the ſaid unknown quantity, and conſequently the value of x, or the root of the original equation, to a greater degree of exactneſs; but that, upon mature conſideration, he had laid it aſide, as tending to introduce new difficulties into the inveſtigation, and to diminiſh it's perſpicuity and ſimplicity. His words are as follows: *An Dominus* de Lagny *librum meum unquam viderat, nec-ne, prorsùs neſcio. Quibus-ce modis non ſolùm ſua* [*ejus*] *methodus, ſed et etiàm aliæ quam plurimæ, eodem prorsùs proceſſu, et perpetuâ inde derivatâ graduum ſcalâ, inveniri poſſint, hujus-ce appendicis eſt oſtendere; idque quàm poſſimus breviſſimè.*

Ipſe equidem de gradatìm inferendis, quas priùs rejeceram in Theoremate Vietæo, *poteſtatibus, olìm cogitavi: ſed tamen non proſecutus fui; utpote qui methodum meam, harum omnium fundamentalem, veluti facillimam ſemper exiſtimavi. Subſequenti proceſſu earum omnium inventionem indagare cuilibet liceat.* See Mr. Raphſon's *Analyſis Æquationum Univerſalis*, edition 2nd, A. D. 1697, page 49. And again, in page 55, he concludes his appendix with theſe words: *Innumeras etiàm alias methodos et abbreviationes,* novarum *quidem* methodorum *nomine inſigniendas, adinvenire liceat; quæ tamen omnia fundamentali huic ſuperiorum poteſtatum, imprimis, rejectionis methodo, poſteáque gradatìm retinendarum, innitantur. Noſtram tamen ſimpliciſſimam fore et facillimam, cuivis pateat.*

OF

OF THE EXTRACTION OF THE SQUARE-ROOT OF A GIVEN NUMBER BY APPROXIMATION.

Art. 55. I HAVE now accompliſhed the object I had principally in view in drawing-up this Appendix to the foregoing Tract of Dr. Halley; which was to make a compariſon between his method of reſolving cubick and other higher affected equations by approximation, which proceeds by the reſolution of quadratick equations, and Mr. Raphſon's method of reſolving them, which proceeds by the reſolution of only ſimple equations. And my readers will now, I hope, be able to determine for themſelves "to which of the two methods they ought to give the preference:" or, if the preference ought in ſome caſes to be given to Dr. Halley's method, and in others to Mr. Raphſon's, to determine "in what caſes Dr. Halley's method, and in what caſes Mr. Raphſon's method, deſerves to be preferred." For my own part, I am decidedly of opinion (as I have declared above in page 27,) that in moſt caſes Mr. Raphſon's method of approximation will be found more convenient than Dr. Halley's, though there may, perhaps, be ſome few occaſions on which it may be expedient to have recourſe to Dr. Halley's method. Now, whenever it ſhall be judged convenient to proceed by Dr. Halley's method, it will evidently be neceſſary, in every ſeparate proceſs of the approximation, to extract the ſquare-root of a given

number;

&c.

number; and in the ſecond and other following proceſſes of ſuch approximations by Dr. Halley's method, it is obvious that the given numbers of which the ſquare-roots muſt be extracted, will be very long numbers, conſiſting often of 12 or 14, or 16 or more, decimal figures. In theſe caſes therefore it may ſeem proper to inquire, whether the labour of extracting theſe ſquare-roots may not be leſſened by applying the method of approximation even to that well-known, and not very difficult, operation, inſtead of performing it in the common way: becauſe in extracting the ſquare-root of a number in the common way we obtain only one new figure of the ſquare-root ſought at every new diviſion of the laſt remainder by the new diviſor formed by doubling the root already found; whereas it may be ſuppoſed that, by ſome of the methods of approximation, we might obtain ſeveral new figures of the root ſought at every new operation, and thereby avoid much unneceſſary labour. Now, in anſwer to this inquiry, I believe, it may ſafely be affirmed, that, in general, it will be found moſt convenient to extract the ſquare-root of a given number in the common way, without having recourſe to any of the methods of approximation; and more eſpecially, when it is propoſed to extract it to only ſix, or ſeven, or even eight, figures: and further, that, if ever it ſhould be judged expedient, when the ſquare-root has already been found, either in the common way, or by means of a table of logarithms, exact to ſeven, or eight, places of figures, to carry the extraction of it to ſeven or eight figures more, or to fifteen or ſixteen figures in all, it will be much better to make uſe of the expreſſion $a + \frac{b}{2a}$, which is afforded

us for this purpoſe by Mr. Raphſon's method of approximation, and is derived from the reſolution of a ſimple equation, than to have recourſe to the expreſſion $a + \frac{ab}{2aa + \frac{b}{2}}$, which is given us above by Dr. Halley in the foregoing tract, page 9, and is derived from the contemplation and imperfect reſolution of a quadratick equation. Theſe things will become evident from the inveſtigation of the two expreſſions $a + \frac{b}{2a}$ and $a + \frac{ab}{2aa + \frac{b}{2}}$, and the application of them to the extraction of the ſquare-root of a particular number; in order to which it will be neceſſary to give a ſolution of the following Problem.

A PROBLEM.

Art. 56. To extract the ſquare-root of a given number by approximation.

SOLUTION.

Let the given number, of which the ſquare-root is to be extracted, and which we will ſuppoſe to be greater than 1, be called N; and let it's unknown ſquare-root be called x.

Then will $x x$ be $=$ N.

Let $a a$ be any ſquare number leſs than N, of which the ſquare-root a is known; and let the exceſs of N above $a a$ be called b.

Then will N be $= a a + b$; and conſequently $x x$, which is $=$ N, will alſo be $= a a + b$; and, therefore, x will be $= \sqrt{a a + b}$.

Now, ſince $a a + b$ is greater than $a a$, it follows that the ſquare-root of $a a + b$ will be greater than the ſquare-root of $a a$; that is, $\sqrt{a a + b}$ will be greater than a. Therefore x will be greater than a.

Further, ſince x is greater than a, let it's exceſs above a be called z.

Then will x be $= a + z$, and conſequently $x x$ will be $(= \overline{a + z}^2 = a a + 2 a z + z z$.

But $x x = a a + b$.

Therefore $a a + 2 a z + z z$ will alſo be $= a a + b$; and conſequently (ſubtracting $a a$ from both ſides,) $2 a z + z z$ will be $= b$.

But z is much leſs than a, and, *à fortiori*, than $2 a$. Therefore $z z$ will be much leſs than $2 a z$, and conſequently $z a z$ alone will be nearly equal to $2 a z + z z$, or to b. And conſequently z will be nearly $= \frac{b}{2a}$,

and

and $a + z$ will be nearly $= a + \frac{b}{2a}$; or $a + \frac{b}{2a}$ will be a near value of $a + z$, or x, or $\sqrt{aa + b}$, or $\sqrt{\text{N}}$. Q. E. I.

This expreſſion $a + \frac{b}{2a}$ is Mr. Raphſon's approximation to the value of $\sqrt{aa + b}$, or $\sqrt{\text{N}}$, or the ſquare-root of the given number N. It is evidently ſomewhat greater than the truth; becauſe $2az + zz$ is $= b$, and conſequently $2az$ is accurately $= b - zz$, and z is accurately $= \frac{b}{2a} - \frac{zz}{2a}$, which is leſs than $\frac{b}{2a}$. But it will uſually give us as many new figures of $a + z$, or the value of $\sqrt{aa + b}$, or $\sqrt{\text{N}}$, or, at leaſt, as many new figures of the ſaid root, wanting one, exactly, as there are figures in a, the part of the root that is already known.

The Inveſtigation of Dr. Halley's Expreſſion for the ſame Purpoſe.

Art. 57. The approximation to the value of $\sqrt{aa + b}$, or $\sqrt{\text{N}}$, given us by Monſieur *de Lagny* and Dr. Halley, is $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, and may be found as follows:

Since $2az + zz$ is $= b$, and $2az + zz$ is $= z \times \overline{2a + z}$, we ſhall have $z \times \overline{2a + z} = b$, and conſequently $z = \frac{b}{\overline{2a + z}}$.

But z is nearly equal to $\frac{b}{2a}$, as we have ſeen in the laſt article.

Therefore $\frac{b}{\overline{2a + z}}$ will be nearly equal to $\frac{b}{2a + \frac{b}{2a}}$, or to $\frac{ab}{2aa + \frac{b}{2}}$, or to $\frac{2ab}{4aa + b}$. Therefore z (which is equal to $\frac{b}{\overline{2a + z}}$,) will be nearly equal to $\frac{b}{2a + \frac{b}{2a}}$, or to $\frac{ab}{2aa + \frac{b}{2}}$, or to $\frac{2ab}{4aa + b}$; and conſequently $a + z$, or $\sqrt{\overline{aa + b}}$, or $\sqrt{\mathrm{N}}$, will be nearly equal to $a + \frac{b}{2a + \frac{b}{2a}}$, or to $a + \frac{ab}{2aa + \frac{b}{2}}$ (which is Dr. Halley's expreſſion,) or to $a + \frac{2ab}{4aa + b}$.

Q. E. I.

Art. 58. This near value of $\sqrt{\overline{aa + b}}$ is always leſs than the truth; becauſe $\frac{b}{2a}$ (which forms the ſecond term

term of the binomial denominator of the fraction $\frac{b}{2a + \frac{b}{2a}}$,) is greater than z, (or the ſecond term of the binomial denominator of the fraction $\frac{b}{2a + z}$,) and conſequently the fraction $\frac{b}{2a + \frac{b}{2a}}$ muſt be leſs than the fraction $\frac{b}{2a + z}$, and $a + \frac{b}{2a + \frac{b}{2a}}$ muſt be leſs than $a + \frac{b}{2a + z}$, or than $a + z$, or $\sqrt{aa + b}$, or $\sqrt{N}$. Q. E. D.

Art. 59. This expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, will approach much nearer than the former expreſſion $a + \frac{b}{2a}$ to the true value of $\sqrt{aa + b}$. But it will be much more difficult to compute than the expreſſion $a + \frac{b}{2a}$, on account of the much greater number of figures in the denominator, or diviſor, $2a + \frac{b}{2a}$, or $2aa + \frac{b}{2}$, or $4aa + b$, than in the denominator, or diviſor, $2a$. And this difficulty is ſo great when a is a number conſiſting of ſeven, or

eight, figures, that I conſider this expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, as of very little uſe in theſe caſes.

A Third Expreſſion for the ſame Purpoſe.

Art. 60. But the ſolution of the foregoing problem will furniſh us with another expreſſion for the value of x, or $\sqrt{aa + b}$, that will approach almoſt as nearly to it's true value as the foregoing expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, given us by Monſieur de Lagny and Dr. Halley, and will be ſomewhat leſs difficult to compute than that expreſſion of Dr. Halley, though much more ſo than Mr. Raphſon's expreſſion $a + \frac{b}{2a}$, and indeed too much ſo to make it worth our while on moſt occaſions to have recourſe to it. This expreſſion is $a + \frac{b}{2a} - \frac{\frac{bb}{2aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3 + 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$, and may be found as follows:

Since $2az + zz$ is $= b$, we ſhall have $2az = b - zz$, and conſequently $z = \frac{b}{2a} - \frac{zz}{2a}$. Therefore,

fore, if we ſubſtitute $\frac{b}{2a}$ (which is nearly equal to, but ſomewhat greater than z) inſtead of z in the term $\frac{zz}{2a}$, we ſhall have z nearly equal to, but ſomewhat greater than, $\frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $\frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $\frac{b}{2a} - \frac{bb}{8a^3}$. Therefore $a + z$ will be nearly equal to, but ſomewhat greater than, $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$; that is, x, or $\sqrt{aa + b}$, will be very nearly equal to, but ſomewhat greater than, the trinomial quantity $a + \frac{b}{2a} - \frac{\frac{bb}{4aa}}{2a}$, or $a + \frac{b}{2a} - \frac{bb}{2a \times 4aa}$, or $a + \frac{b}{2a} - \frac{bb}{8a^3}$. Q. E. I.

Note. The three terms of this laſt expreſſion $a + \frac{b}{2a} - \frac{bb}{8a^3}$ are the ſame with the three firſt terms of the infinite ſeries for expreſſing the value of $\sqrt{aa+b}$, or $\overline{aa + b}^{\frac{1}{2}}$, derived from Sir Iſaac Newton's binomial theorem.

Art. 61. We will now apply theſe different expreſſions to the inveſtigation of the ſquare-root of the number 32, or 25 + 7, (in which 25 anſwers to $a\,a$, and 7 to b, in the foregoing general expreſſion $a\,a + b$,) in order to try how far they will contribute to facilitate the extraction of that ſquare-root. And that the advantage ariſing from the uſe of them above the common method of extracting the ſquare-root (if there is any ſuch advantage,) may appear the more clearly, I will firſt exhibit the operation of extracting this ſquare-root to thirteen places of figures by the common method. Now this extraction will be as follows:

The

The Extraction of the Square-Root of the Number 32 *by the Common Method.*

32 (5.656,854,249,492,
25

106) ·7.00
636

1125) ·6400
5625

11306) ·77500
67836

113128) ·966400
905024

1131365) ·6137600
5656825

11313704) ·4807500
4525481 6

113137082)·282268400
226274164

1131370844)·5599423600
4525483376

11313708489) 1073940224 00
1018233764 01

113137084984)··557064599900
452548339936

1131370849889) 1045162599640 0
1018233764900 1

113137084998 2)··26928834739900
22627416997964

·4301417741936

The

The Extraction of the Square-Root of the Number 32 *by Approximation by means of the Expreſſion* $a + \frac{b}{2a}$ *given us for that purpoſe by Mr. Raphſon.*

Art. 62. Let it now be required to find a near value of the ſquare-root of the number 32, or 25 + 7, by means of the expreſſion $a + \frac{b}{2a}$ found above, in art. 56, by Mr. Raphſon's method of approximation.

Here in the 1ſt place we ſhall have $aa = 25$, and $b = 7$; and therefore a will be $= 5$, and $\frac{b}{2a}$ will be $(= \frac{7}{2 \times 5} = \frac{7}{10}) = 0.7$. Therefore $a + \frac{b}{2a}$ will be $(= 5 + 0.7) = 5.7$; or the ſquare-root of $aa + b$, or 25 + 7, or 32, will be nearly equal to, but ſomewhat leſs than, 5.7. Q. E. I.

Secondly, ſince $\sqrt{32}$ is leſs than 5.7, let us ſuppoſe it to be nearly equal to 5.6, and let us raiſe 5.6 to it's ſquare, in order to diſcover whether the ſaid ſquare will be greater, or leſs, than 32, and conſequently whether 5.6 is greater, or leſs, than $\sqrt{32}$.

Now the ſquare of 5.6 is 31.36; which is leſs than 32. Therefore 5.6 muſt be leſs than the ſquare-root of 32.

We

We will therefore now take $a = 5.6$. And we ſhall then have $aa = 31.36$, and conſequently b ($= 32.00 - 31.36$) 0.64, and $\frac{b}{2a}$ ($= \frac{0.64}{2 \times 5.6} = \frac{0.64}{11.2} =$ 0.057, and $a + \frac{b}{2a}$ ($= 5.6 + 0.057$) $= 5.657$. Therefore 5.657 will be a third near value of the ſquare-root of ($aa + b$, or $31.36 + 0.64$, or) the propoſed number 32. Q. E. I.

Now let 5.657 be raiſed to it's ſquare in order to diſcover whether the ſaid ſquare will be greater, or leſs, than 32, and conſequently whether 5.657 itſelf will be greater, or leſs, than the true value of $\sqrt{32}$.

Now $\overline{5.657}^2$ is $= 32.001,649$; which is a little greater than 32. Therefore 5.657 is a little greater than the true value of $\sqrt{32}$. We will therefore ſuppoſe $\sqrt{32}$ to be between 5.657 and 5.656, and will raiſe 5.656 to it's ſquare, in order to diſcover whether the ſaid ſquare will be greater, or leſs, than 32, and conſequently whether 5.656 will be greater, or leſs, than $\sqrt{32}$.

Now the ſquare of 5.656 is $= 31.990,336$; which is leſs than 32; and conſequently 5.656 muſt be leſs than $\sqrt{32}$.

We will therefore now, in the 3d place, make $a = 5.656$; and we ſhall then have $aa = 31.990,336$, and b (=

b (= 32.000,000 − aa = 32.000,000 − 31.990,336) = 0.009,664. Therefore $\frac{b}{2a}$ will be (= $\frac{0.009,664}{2 \times 5.656}$ = $\frac{0.009,664}{11.312}$) = 0.000,854, and conſequently $a + \frac{b}{2a}$ will be (= 5.656 + 0.000,854) = 5.656,854. Therefore 5.656,854 will be a fourth near value of the ſquare-root of 32. Q. E. I.

This fourth near value of the ſquare-root of 32 is very nearly equal to, but ſomewhat leſs than, it's true value. For the ſquare of 5.656,854 is = 31.999,997,177,316, which is leſs than 32 by the very ſmall number 0.000,002,822,684.

Note. If we had carried the diviſion of b by $2a$, or of 0.009,664 by 11.312, to two figures more in the quotient, the ſaid quotient would have been 0.000,854,31, and conſequently $a + \frac{b}{2a}$ would have been (= 5.656, + 0.000,854,31) = 5.656,854,31, which would have been greater than the true value of the ſquare-root of 32, or $aa + b$, agreeably to what is ſhewn above in art. 56. But by carrying the diviſion to only the three ſignificant figures 0.000,854, of the quotient, the value of $a + \frac{b}{2a}$ is kept under the true value of $\sqrt{32}$.

Since 5.656,854 is leſs than the true value of $\sqrt{32}$, we will, in order to obtain a nearer value of the ſaid ſquare-

ſquare-root, take $a = 5.656,854$. And we ſhall then have $a\,a = 31.999,997,177,316$, and b $(= 32 - a\,a = 32.000,000,000,000 - 31.999,997,177,316) = 0.000,002,822,684$. Therefore $\frac{b}{2a}$ will be $(= \frac{0.000,002,822,684}{2 \times 5.656,854} = \frac{0.000,002,822,684}{11.313,708}) = 0.000,000,249,492$, and $a + \frac{b}{2a}$ will be $(= 5.656,854 + 0.000,000,249,492) = 5.656,854,249,492$; that is, the fifth near value of the ſquare-root of the number 32, obtained by this fourth proceſs of Mr. Raphſon's method of approximation, will be 5.656,854,249,492. Q. E. I.

All the figures of this number 5.656,854,249,492 are exact, and agree with the figures of the ſquare-root of 32 as obtained above in art. 61. by the common method of extracting it. But it appears to me that, from the neceſſity of ſquaring the ſeveral ſucceſſive values of a, in order to obtain the value of b, or $32 - a\,a$, and more eſpecially that of ſquaring the fourth value 5.656,854, (which conſiſts of ſeven figures,) and from the number of diviſions to be gone through in order to obtain the ſucceſſive values of $\frac{b}{2a}$, and likewiſe from the variety of new ſuppoſitions made in this method, and the reaſonings conſequent upon them, there is more time and trouble employed in obtaining theſe thirteen figures 5.656,854,249,492 of the ſquare-root of the propoſed number 32 by this method of approximation than there is in obtaining them by the common method of extracting the ſquare-root in the manner exhibited above in art. 61.

I will

b (= 32.000,000 −

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But by carrying the diviſion to only the
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is kept under the true value of $\sqrt{32}$.

Since 5.656,854 is leſs than the true
we will, in order to obtain a nearer v

I will now proceed to extract the ſquare-root of this ſame number 32 by means of the expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, or $a + \frac{ab}{2aa + \frac{b}{2}}$, or $a + \frac{2ab}{4aa + b}$, given us by Monſieur de Lagny and Dr. Halley for the near value of the ſquare-root of $aa + b$.

The Extraction of the Square-Root of the Number 32 *by Approximation, by means of the Expreſſion given us for that purpoſe by Monſieur de Lagny and Dr. Halley, and inveſtigated above in Art.* 57.

Art. 63. Of the three equivalent expreſſions $a + \frac{b}{2a + \frac{b}{2a}}$, $a + \frac{ab}{2aa + \frac{b}{2}}$, and $a + \frac{2ab}{4aa + b}$, found above in art. 57, for a near value of $\sqrt{aa + b}$, I take the firſt expreſſion, $a + \frac{b}{2a + \frac{b}{2a}}$, to be the fitteſt for calculation; becauſe it does not require our multiplying b into a, in order to obtain the quantity ab, which occurs in the numerators of the fractions $\frac{ab}{2aa + \frac{b}{2}}$ and $\frac{2ab}{4aa + b}$ in the two other expreſſions. I ſhall therefore here compute the near value of the ſquare-

ſquare-root of 32, or 25 + 7, or $a\,a + b$, by means of the ſaid firſt expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$.

Now, ſince $a\,a$ is = 25, and b is = 7, we ſhall have a = 5, and $\frac{b}{2a}$ $(= \frac{7}{2 \times 5} = \frac{7}{10}) = 0.7$, and $2a + \frac{b}{2a}$ $(= 2 \times 5 + 0.7 = 10 + 0.7) = 10.7$, and conſequently $\frac{b}{2a + \frac{b}{2a}}$ $(= \frac{7}{10.7}) = 0.65$. Therefore $a + \frac{b}{2a + \frac{b}{2a}}$ will be $(= 5 + 0.65) = 5.65$; that is, the ſecond near value of the ſquare-root of 32, (obtained by means of the expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, after ſuppoſing it's firſt near value to be a, or 5,) will be 5.65; of which all the figures are exact. Q. E. I.

This value we know to be leſs than the truth. Therefore, to obtain a third near value of $\sqrt{32}$, we will make uſe of the ſame expreſſion as before, to wit, $a + \frac{b}{2a + \frac{b}{2a}}$, only making a (which before ſtood for 5, or the firſt near value of $\sqrt{32}$,) now ſtand for 5.65, or the ſecond near value $\sqrt{32}$, that has been juſt now obtained.

Now,

Now, if a is $= 5.65$, we ſhall have $aa = (\overline{5.65}|^2) = 31.9225$, and $b\ (= 32 - aa = 32.0000 - 31.9225) = 0.0775$. Therefore $\frac{b}{2a}$ will be $(= \frac{0.0775}{2 \times 5.65} = \frac{0.0775}{11.30}) = 0.006,858$, and $2a + \frac{b}{2a}$ will be $(= 11.30 + 0.006,858) = 11.306,858$, and $\frac{b}{2a + \frac{b}{2a}}$ will be $(= \frac{0.0775}{11.306,858}) = 0.006,854,24$. Therefore $a + \frac{b}{2a + \frac{b}{2a}}$ will be $(= 5.65 + 0.006,854,24) = 5.656,854,24$; that is, the third near value of the ſquare-root of the number 32, which has been obtained by means of the expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, will be 5.656,854,24; which is exact in all it's nine figures.

Q. E. I.

We might now proceed to compute a fourth near value of this ſquare-root by means of a third application of the expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, by putting $a = 5.656,854,24$, and ſquaring the ſaid number in order to obtain the value of aa, and then ſubtracting the ſaid ſquare from 32, in order to obtain the value of $N - aa$, or $32 - aa$, or b; and by, then, dividing b by $2a$, or $2 \times 5.656,854,24$, or by $11.313,708,48$, and adding the

the quotient, (continued to eight, or nine, figures,) to $2a$, or 11.313,708,48, and dividing b by the ſum thence ariſing, and adding the quotient of this laſt diviſion to a, or 5.656,854,24. And, if we were to do ſo, the near value of $\sqrt{32}$ thereby obtained would be exact to 27, or 26, figures. But the labour of the calculation would be very great; partly on account of the ſquaring of the long number 5.656,854,24, and partly on account of the diviſion of b by $2a + \frac{b}{2a}$, which, it is obvious, would be a moſt tedious operation, becauſe that diviſor would be a very long number, conſiſting of 17 or 18 figures. And therefore I conſider the further approximation to the value of the ſquare-root of 32 by means of this expreſſion $a + \frac{b}{2a + \frac{b}{2a}}$, as being perfectly inexpedient, and, in a manner, impracticable.

Art. 64. It remains that we ſhould inveſtigate the ſquare-root of the ſame number 32 by means of the third expreſſion $a + \frac{b}{2a} - \frac{bb}{8a^3}$, which co-incides with the three firſt terms of the infinite ſeries for expreſſing the value of $\sqrt{aa + b}$, or $\overline{aa + b}|^{\frac{1}{2}}$, derived from Sir Iſaac Newton's binomial theorem. Now this may be done as follows:

The Extraction of the Square-Root of the Number 32 *by Approximation by means of the Trinomial Expreſſion* $a + \frac{b}{2a} - \frac{bb}{8a^3}$.

If a is at firſt taken equal to 5, (as before,) and conſequently $aa = 25$, and $b\ (= 32 - aa = 32 - 25) = 7$, we ſhall have $a + \frac{b}{2a} - \frac{bb}{8a^3}\ (= 5 + \frac{7}{2 \times 5} - \frac{49}{8 \times 125} = 5 + \frac{7}{10} - \frac{49}{1000} = 5 + 0.7 - 0.049 = 5.700 - 0.049) = 5.651$; that is, 5.651, or (dropping the laſt figure 1,) 5.65, will be the ſecond near value of $\sqrt{aa + b}$, or $\sqrt{25 + 7}$, or $\sqrt{32}$, that is obtained by this firſt application of the trinomial expreſſion $a + \frac{b}{2a} - \frac{bb}{8a^3}$. Q. E. I.

Now let a be put $= 5.65$. And we ſhall then have $aa\ (= \overline{5.65}^2) = 31.9225$, and $b\ (= 32.0000 - 31.9225) = 0.0775$. Therefore $\frac{b}{2a}$ will be $(= \frac{0.0775}{2 \times 5.65} = \frac{0.0775}{11.30}) = 0.006,858,40$, and $a + \frac{b}{2a}$ will be $(= 5.65 + 0.006,858,40) = 5.656,858,40$; and $\frac{bb}{4aa}$ will be $(= \frac{b}{2a} \times \frac{b}{2a} = \overline{0.006,858,40}^2) =$

$= 0.000,047,037,650,560,0$, and $\frac{bb}{8a^3}$ will be $(= \frac{bb}{2a \times 4aa} = \frac{0.000,047,037,650,560,0}{11.30}) = 0.000,004,16$ &c; and consequently $a + \frac{b}{2a} - \frac{bb}{8a^3}$ will be $(= 5.656,858,40 - 0.000,004,16$ &c$) = 5.656,854,24$; that is, the third near value of the square-root of 32, which is obtained by this second application of the trinomial expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$, will be 5.656, 854,24; of which number all the nine figures are exact. Q. E. I.

If we were to carry this approximation one step further, by taking $a = 5.656,854,24$, and computing aa and $32 - aa$, or b, and the trinomial expression $a + \frac{b}{2a} - \frac{bb}{8a^3}$, or $a + \frac{b}{2a} - \frac{bb}{\frac{4aa}{2a}}$, we should obtain the value of the square-root of 32 exact to 26 or 27, figures. But the labour of computing the second and third terms $\frac{b}{2a}$ and $\frac{bb}{4aa}$, of this expression to that number of figures would be very great, though, I believe, not quite so great as that of computing $\frac{b}{2a + \frac{b}{2a}}$, (or the second term of the expression $a + \frac{b}{2a + \frac{b}{2a}}$ given by Monsieur de Lagny and

Dr. Halley for the ſame purpoſe.) to the ſame number of figures. I ſhould therefore conſider both theſe expreſſions in this caſe as unfit for practice, and ſhould think it would be much more adviſeable, if the ſquare-root of 32 was required to be found exactly to more than the nine figures 5.656,854,24 already diſcovered, to find the next nine figures of it's value either by computing Mr. Raphſon's much ſimpler and eaſier expreſſion $a + \frac{b}{2a}$, or by continuing the extraction of it by the common method to that extent.

Art. 65. And, indeed, from all theſe trials of theſe different expreſſions for approximating to the value of the ſquare-root of a given number N, or $aa + b$, I am inclined to conclude that, in performing the extractions of the ſquare-roots of given numbers which are neceſſary in Dr. Halley's method of reſolving high affected equations, it will almoſt always be found eaſier and more convenient to proceed by the common method of extracting them than to have recourſe to either of the three expreſſions above mentioned, not excepting even Mr. Raphſon's ſimple expreſſion $a + \frac{b}{2a}$.

End of the Obſervations on the Extraction of the Square-Root of a given Number by the Methods of Approximation.

A ME-

A METHOD OF DISCOVERING WHETHER THE THREE EQUATIONS THAT HAVE BEEN RESOLVED IN THE FOREGOING ARTICLES HAVE ANY OTHER ROOTS BESIDES THOSE ABOVE INVESTIGATED.

Of the Number of Roots in the Cubick Equation $x^3 - 17x^2 + 54x = 350$.

Art. 66. THE firſt of the three equations that have been reſolved in the foregoing articles is the cubick equation $x^3 - 17x^2 + 54x = 350$, of which we have found one root to be = 14.954,068,61. Now, as there are two changes of the ſigns + and — prefixed to the terms that form the left-hand ſide of this equation, it may be ſuſpected that there may be either one or two more roots to it, or values of x that, being ſubſtituted inſtead of x in the trinomial quantity $x^3 - 17x^2 + 54x$, will make the ſaid quantity be equal to 350, or the abſolute term of the ſaid equation. This therefore is a matter which ought to be inquired into and determined, before the reſolution of the equation can be conſidered as quite compleat.

Art. 67. Now, if this equation has another root, that root muſt be either greater or leſs than 14.954,068,61, or (omitting all the figures of this root except the three firſt, 14.9, in order to leſſen the labour of the following calculations,) leſs than 14.9. Let it firſt be ſuppoſed to be greater than 14.9, and to be called c; and let b be put = 14.9.

Then we ſhall have $b^3 - 17b^2 + 54b = 350$, and alſo $c^3 - 17c^2 + 54c = 350$; and conſequently $c^3 - 17c^2 + 54c$ will be $= b^3 - 17b^2 + 54b$. Therefore (adding $17c^2$ to both ſides) we ſhall have $c^3 + 54c = b^3 + 17c^2 - 17b^2 + 54b$, and (ſubtracting $b^3 + 54b$ from both ſides) $c^3 - b^3 + 54c - 54b = 17c^2 - 17b^2$, or $c^3 - b^3 + 54 \times \overline{c - b} = 17 \times \overline{c^2 - b^2}$. Therefore $\frac{c^3 - b^3}{c - b} + 54 \times \frac{c - b}{c - b}$ will be $= 17 \times \frac{\overline{c^2 - b^2}}{c - b}$, or $c^2 + cb + b^2 + 54$ will be $= 17 \times \overline{c + b} = 17c + 17b$. Therefore (ſubſtituting 14.9 inſtead of b,) $c^2 + 14.9 \times c + \overline{14.9}^2 + 54$ will be $= 17c + 17 \times 14.9$, that is, $c^2 + 14.9 \times c + 222.01 + 54$ will be $= 17c + 253.3$. Therefore (ſubtracting 222.01 from both ſides,) we ſhall have $c^2 + 14.9 \times c + 54 = 17c + 31.29$, and (ſubtracting 31.29 from both ſides) $c^2 + 14.9 \times c + 22.71 = 17c$, and (ſubtracting $14.9c$ from both ſides,) $c^2 + 22.71 = 2.1 \times c$; that is, the ſum of c^2 and 22.71 is equal to $2.1 \times c$. Therefore c^2 alone muſt be leſs than $2.1 \times c$, and conſequently (dividing both ſides by c,) c muſt be leſs than 2.1. But c was ſuppoſed to be greater than b, or 14.9, which is much greater than 2.1. Therefore c or the ſuppoſed ſecond value of x in the equation $x^3 - 17x^2 - 54x = 350$ will be both greater and leſs than 2.1; which is impoſſible. And this impoſſible concluſion followed from the ſuppoſition that the equation $x^3 - 17x^2 + 54x = 350$ had another root c that was greater than b,

or 14.9. Therefore that ſuppoſition was a falſe ſuppoſition, and the ſaid equation cannot have any other root that is greater than 14.9.

Art. 68. It remains that we inquire whether the ſaid equation cannot have another root beſides b, or 14.9, that ſhall be leſs than 14.9.

Now, if this be poſſible, let the ſaid root, that is leſs than b, or 14.9, be called a.

Then we ſhall have $b^3 - 17b^2 + 54b = 350$, and $a^3 - 17a^2 + 54a = 350$, and conſequently $b^3 - 17b^2 + 54b = a^3 - 17a^2 + 54a$. Therefore, (adding $17b^2$ to both ſides,) we ſhall have $b^3 + 54b = a^3 + 17b^2 - 17a^2 + 54a$, and (ſubtracting $a^3 + 54a$ from both ſides,) $b^3 - a^3 + 54b - 54a = 17b^2 - 17a^2$, or $b^3 - a^3 + 54 \times \overline{b - a} = 17 \times \overline{b^2 - a^2}$. Therefore $\frac{b^3 - a^3}{b - a} + 54 \times \frac{b - a}{b - a}$ will be $= 17 \times \frac{b^2 - a^2}{b - a}$; that is, $b^2 + ba + aa + 54$ will be $= 17 \times \overline{b + a} = 17b + 17a$; or (ſubſtituting 14.9 inſtead of b in this equation) $\overline{14.9}^2 + 14.9 \times a + aa + 54$ will be $= 17 \times 14.9 + 17a$, or $222.01 + 14.9 \times a + aa + 54$ will be $= 253.30 + 17a$. Therefore (ſubtracting 222.01 from both ſides,) we ſhall have $14.9 \times a + aa + 54 = 31.29 + 17a$, and (ſubtracting 31.29 from both ſides,) $14.9 \times a + aa + 22.71 = 17a$, and (ſubtracting $14.9 \times a$ from both ſides,) $aa + 22.71 = 2.1 \times a$; and, laſtly, (ſubtracting aa from both ſides,) $2.1 \times a - aa = 22.71$, or $\overline{2.1 - a} \times a =$ 22.71;

22.71; that is, the product of the multiplication of $2.1 - a$ into a will be equal to 22.71. But the greateſt poſſible product of the multiplication of $2.1 - a$ into a is the ſquare of half 2.1, or the ſquare of 1.05, which is 1.1025. Therefore it is impoſſible that the binomial quantity $2.1\,a - a\,a$ ſhould be equal to 22.71. And this impoſſible concluſion has been regularly deduced from the ſuppoſition that the equation $x^3 - 17\,x^2 + 54\,x = 350$ had another root beſides 14.9, that was leſs than 14.9. Therefore that ſuppoſition was a falſe ſuppoſition, and the ſaid equation $x^3 - 17\,x^2 + 54\,x = 350$ cannot have any root beſides 14.9, that is leſs than 14.9.

But it was ſhewn before, that the ſaid equation cannot have any root beſides 14.9 that is greater than 14.9.

Therefore the ſaid equation cannot have any other root whatſoever beſides 14.9, or 14.954,068,61; and therefore this number will be it's only root. Q. E. I.

Another Method of proving that the Equation $x^3 - 17\,x^2 + 54\,x = 350$ *has no other Root but the Root* 14.954,068,61, *that has been already found.*

Art. 69. In the two foregoing articles it has been ſhewn that the equation $x^3 - 17\,x^2 + 54\,x = 350$ cannot have any other root beſides 14.954,068,61, or 14.9, by proving that, if it were ſuppoſed to have any other root, either greater or leſs than 14.9, an impoſſible concluſion would follow from ſuch ſuppoſition. This indirect way of proving a propoſition is not quite ſo ſatisfactory, or, at leaſt, ſo pleaſing to the mind, as a direct

direct proof. I will therefore now endeavour to prove the same thing in a direct manner, by shewing, 1st, that, if we suppose x to increase from 14.9, or b, to any greater quantity c, which exceeds 14.9, or b, by the difference d, the sum of the increments received by the trinomial quantity $x^3 - 17x^2 + 54x$, or $b^3 - 17b^2 + 54b$, during the increase of x from b to c, or $b + d$, will be greater than the sum of the decrements which it will suffer, or the sum of the quantities that will be subtracted from it, and consequently that the said trinomial quantity $x^3 - 17x^2 + 54x$ (which will then be equal to $c^3 - 17c^2 + 54c$, or $\overbrace{b+d}^3 - 17 \times \overbrace{b+d}^2 + 54 \times \overline{b+d}$) will, upon the whole, be greater than it was before, or than the absolute term 350, and consequently that c, or $b + d$, will not be a root of the equation $x^3 - 17x^2 + 54x = 350$; and by shewing, 2ndly, that, if we suppose x to decrease from 14.9, or b, to any lesser quantity a, which falls short of b by the difference d, the sum of the increments received by the trinomial quantity $x^3 - 17x^2 + 54x$, or $b^3 - 17b^2 + 54b$, during the decrease of x from b to a, or $b - d$, will be less than the sum of the decrements it will suffer, or the sum of the quantities that will be subtracted from it, and consequently that the said trinomial quantity $x^3 - 17x^2 + 54x$ (which will then be equal to $a^3 - 17a^2 + 54a$, or $\overbrace{b-d}^3 - 17 \times \overbrace{b-d}^2 + 17 \times \overline{b-d}$,) will, upon the whole, be less than it was before, or than the absolute term 350, and consequently that a, or $b - d$, will not be a root of

of the equation $x^3 - 17x^2 + 54x = 350$. Theſe things may be proved in the manner following:

Art. 70. If x is ſuppoſed to increaſe from b, or $14.\dot{9}$, to c, or $b + d$, the trinomial quantity $x^3 - 17x^2 + 54x$, will be changed from $b^3 - 17b^2 + 54b$ to $c^3 - 17c^2 + 54c$, or $\overline{b+d}^3 - 17 \times \overline{b+d}^2 + 54 \times \overline{b+d}$, or $b^3 + 3b^2d + 3bd^2 + d^3 - 17 \times \overline{b^2 + 2bd + d^2} + 54 \times \overline{b+d}$, or $b^3 + 3b^2d + 3bd^2 + d^3 - 17b^2 - 34bd - 17d^2 + 54b + 54d$, or

$$\left\{\begin{array}{l} b^3 + 3b^2d + 3bd^2 + d^3 \\ -17b^2 - 34bd - 17d^2 \\ +54b + 54d \end{array}\right\}; \text{ which}$$

contains the trinomial quantity $b^3 - 17b^2 + 54b$ (which is equal to 350) together with the four quantities $3b^2d$, $3bd^2$, d^3, and $54d$, which are marked with the ſign +, or are added to $b^3 - 17b^2 + 54d$, and conſequently tend to increaſe it, and the two quantities $34bd$ and $17d^2$, which are marked with the ſign −, or are ſubtracted from $b^3 - 17b^2 + 54b$, and conſequently tend to diminiſh it. We muſt therefore demonſtrate that the ſum of the four quantities $3b^2d$, $3bd^2$, d^3, and $54d$ will be greater than the ſum of the two quantities $34bd$ and $17d^2$.

Art. 71. Now the quadrinomial quantity $3b^2d + 3bd^2 + d^3 + 54d$ is $(= 3 \times \overline{14.9}^2 \times d + 3 \times$

14.9

$14.9 \times d^2 + d^3 + 54d = 3 \times 222.01 \times d + 44.7 \times d^2 + d^3 + 54d = 666.03 \times d + 44.7 \times d^2 + d^3 + 54d) = 720.03 \times d + 44.7 \times d^2 + d^3$; and the binomial quantity $34bd + 17d^2$ is $\{= 34 \times 14.9 \times d + 17d^2) = 506.6 \times d + 17d^2$.

But $720.03 \times d$ is greater than $506.6 \times d$, and $44.7 \times d^2$ is greater than $17d^2$. Therefore the binomial quantity $720.03 \times d + 44.7 \times d^2$ will be greater than the binomial quantity $506.6 \times d + 17d^2$. Therefore, *à fortiori*, the trinomial quantity $720.03 \times d + 44.7 \times d^2 + d^3$ will be greater than the binomial quantity $506.6 \times d + 17d$. Therefore the quadrinomial quantity $3b^2d + 3bd^2 + d^3 + 54d$ (which is equal to the trinomial quantity $720.03 \times d + 44.7 \times d^2 + d^3$,) will be greater than the binomial quantity $506.6 \times d + 17d$, and consequently than it's equal, the binomial quantity $34bd + 17d^2$. Therefore the trinomial quantity $\overline{b+d}|^3 - 17 \times \overline{b+d}|^2 + 54 \times b + d$ will be greater than the trinomial quantity $b^3 - 17b^2 + 54b$, or the absolute term 350; and consequently c, or $b + d$, cannot be a root of the proposed equation $x^3 - 17x^2 + 54x = 350$.

Q. E. D.

Art. 72. Secondly, if x is supposed to decrease from b, or 14.9, to a, or $b - d$, the trinomial quantity $x^3 - 17x^2 + 54x$ will be changed from $b^3 - 17b^2 + 54b$ to $a^3 - 17a^2 + 54a$, or $\overline{b-d}|^3 - 17 \times \overline{b-d}|^2 + 54 \times \overline{b-d}$, or $b^3 - 3b^2d + 3bd^2 - d^3 - 17 \times$

b^2

$\overline{b^2 - 2bd + d^2} + 54 \times \overline{b - d}$, or $b^3 - 3b^2d + 3bd^2 - d^3 - 17b^2 + 34bd - 17d^2 + 54b - 54d$, or

$$\left\{\begin{array}{l} b^3 - 3b^2d + 3bd^2 - d^3 \\ -17b^2 + 34bd - 17d^2 \\ +54b - 54d \end{array}\right\};$$ which

contains the trinomial quantity $b^3 - 17b^2 + 54b$ (which is equal to 350,) together with the four quantities $3b^2d$, d^3, $17d^2$, and $54d$, which are marked with the sign —, or subtracted from the quantity $b^3 - 17b^2 + 54b$, and consequently tend to diminish it, and the two quantities $3bd^2$ and $34bd$, which are marked with the sign +, or are added to the quantity $b^3 - 17b^2 + 54b$, and consequently tend to increase it. We must therefore demonstrate that the sum of the four quantities $3b^2d$, d^3, $17d^2$, and $54d$, which are marked with the sign —, is greater than the sum of the two quantities $3bd^2$ and $34bd$, which are marked with the sign +.

Art. 73. Now the quadrinomial quantity $3b^2d + d^3 + 17d^2 + 54d$ is $(= 3 \times \overline{14.9}^2 \times d + d^3 + 17d^2 + 54d = 3 \times 222.01 \times d + d^3 + 17d^2 + 54d = 666.03 \times d + d^3 + 17d^2 + 54d) = 720.03 \times d + 17d^2 + d^3$; and the binomial quantity $3bd^2 + 34bd$ is $= 3 \times 14.9 \times d^2 + 34 \times 14.9 \times d) = 44.7 \times d^2 + 506.6 \times d$. Therefore, if we can shew that the trinomial quantity $720.03 \times d + 17d^2 + d^3$ is greater than the binomial quantity $44.7 \times d^2 +$

506.6

506.6 × d, or 506.6 × d + 44.7 × d^2, it will follow that the quadrinomial quantity $3b^2d + d^3 + 17d^2 + 54$ will be greater than the binomial quantity $3bd^2 + 34bd$, and conſequently that the trinomial quantity $a^3 - 17a^2 + 54a$, or $\overline{b-d}|^3 - 17 \times \overline{b-d}|^2 + 54 \times \overline{b-d}$, will be leſs than the trinomial quantity $b^3 - 17b^2 + 54b$, or than the abſolute term 350 of the propoſed equation $x^3 - 17x^2 + 54x = 350$, and conſequently that a, or $b - d$, cannot be a root of the ſaid equation.

Art. 74. Now that the trinomial quantity 720.03 × d + 17 d^2 + d^3 muſt always be greater than the binomial quantity 506.6 × d + 44.7 × d^2 may be demonſtrated as follows:

The trinomial quantity 720.03 × d + 17d^2 + d^3 will be greater than the binomial quantity 506.6 × d + 44.7 × d^2 if the quotient of the diviſion of the former quantity by d is greater than the quotient of the diviſion of the latter quantity by d, that is, if the trinomial quantity 720.03 + 17d + d^2 is greater than the binomial quantity 506.6 + 44.7 × d. We muſt therefore demonſtrate that the trinomial quantity 720.03 + 17d + dd is greater than the binomial quantity 506.6 + 44.7 × d.

Now the trinomial quantity 720.03 + 17d + dd will be greater than the binomial quantity 506.6 + 44.7 × d, if 720.03 − 506.60 + 17d + dd is greater than

than $44.7 \times d$, or if $720.03 - 506.60 + dd$ is greater than $44.7 \times d - 17d$, or if $213.43 + dd$ is greater than $27.7 \times d$, or if 213.43 is greater than $27.7 \times d - dd$, or than $\overline{27.7 - d} \times d$. But the greateſt poſſible magnitude of $\overline{27.7 - d} \times d$ is the ſquare of half 27.7, or the ſquare of 13.85, which is leſs than the ſquare of 14, or than 196. Therefore 213.43 (which is greater than 196,) muſt be greater than $\overline{27.7 - d} \times d$, or than $27.7 \times d - dd$; and conſequently the trinomial quantity $720.03 + 17d + dd$ will be greater than the binomial quantity $506.6 + 44.7 \times d$, and the trinomial quantity $720.03 \times d + 17d^2 + d^3$ will be greater than the binomial quantity $506.6 \times d + 44.7 + dd$.

Q. E. D.

It appears therefore that neither $b + d$, nor $b - d$, that is, that no quantity either greater or leſs than b, or 14.9, or 14.954,068,61, can be a root of the propoſed equation $x^3 - 17x^2 + 54x = 350$.

Of the Number of Roots in the Biquadratick Equation $x^4 - 3xx + 75x = 10{,}000$.

Art. 75. We will next examine the ſecond equation above-reſolved, to wit, the biquadratick equation $x^4 - 3x^2 + 75x = 10{,}000$, of which we have found one root to be $= 9.886{,}002{,}70$, and will proceed to inquire whether the ſaid equation has, or has not, any other root, or whether there is any other quantity beſides 9.886,002,70, which, being ſubſtituted inſtead of x in the trinomial quantity $x^4 - 3x^2 + 75x$, will make that quantity be equal to the abſolute term 10,000.

Now

Now this inquiry may be made with reſpect to this equation with great eaſe and brevity in the following manner:

If x is ſuppoſed to increaſe from 0 *ad infinitum*, it is evident that, while x is leſs than $\sqrt{3}$, and xx is conſequently leſs than 3, x^4 will be leſs than $3xx$, and conſequently the whole trinomial quantity $x^4 - 3xx + 75x$ muſt be leſs than $75x$. But, while x is leſs than $\sqrt{3}$, $75x$ will be leſs than $75 \times \sqrt{3}$, or 75×1.732, &c, or 129.9. Therefore, while x is leſs than $\sqrt{3}$, the whole trinomial quantity, (which is leſs than $75x$,) will, *à fortiori*, be leſs than 129.9, and therefore will be much leſs than 10,000, or the abſolute term of the equation $x^4 - 3xx + 75x = 10{,}000$. Therefore it is plain that no quantity leſs than 1.732, or $\sqrt{3}$, can be a root of this equation. And, when x is greater than $\sqrt{3}$, and xx is greater than 3, it is evident that x^4 muſt be greater than $3xx$, and that, while xx increaſes from 3 *ad infinitum*, the compound quantity $x^4 - 3xx$ (which is $= xx \times \overline{xx - 3}$) will increaſe continually at the ſame time from 0 *ad infinitum*, and the third term $75x$ will likewiſe increaſe continually from $75 \times \sqrt{3}$ *ad infinitum*. Therefore the ſum of theſe two quantities $x^4 - 3xx$ and $75x$, that is, the trinomial quantity $x^4 - 3xx + 75x$, will likewiſe increaſe from ($0 + 75 \times \sqrt{3}$, or from) $75 \times \sqrt{3}$ *ad infinitum* without ever decreaſing, and therefore will once, and but once, become equal to any propoſed quantity whatſoever that is greater than $75 \times \sqrt{3}$, and conſequently will once, and but once, become equal to 10,000, or the abſolute

term

term of the equation $x^4 - 3xx + 75x = 10{,}000$; or, in other words, that equation will have one, and but one, root. Q. E. I.

This eaſy method of proving that this equation can have only one root was ſuggeſted to me by the ingenious *Mr. William Frend*, Fellow of Jeſus College, Cambridge, and author of a valuable Treatiſe on Algebra in one volume, octavo, intitled *Principles of Algebra*, in which he has explained the ſubject in a very clear and ſcientifick manner, and without any mention of the obſcure and uſeleſs doctrine of negative quantities, or quantities leſs than nothing, or quantities obtained by ſubtracting a greater quantity from a leſſer.

Of the Number of Roots in the Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$.

Art. 76. We will laſtly examine the third equation above-reſolved, to wit, the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, and inquire, whether, or not, it has any other roots beſides the root above found, to wit, 12.756,441,794,480,744,022.

Now let b be put for the root 12.756, &c that is already found, or, rather, for the whole number 13, (which is nearly equal to it,) in order to facilitate the calculations which we ſhall have occaſion to make in the courſe of our inquiry. And let us then ſuppoſe x to increaſe from 13, or b, to ſome greater quantity which we will call c, and which ſhall exceed the known root 13, or b, by the difference d.

Then

Then it is evident that, when x is increaſed from 13, or b, to c, or $b + d$, the quadrinomial quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$ (which is equal to the abſolute term 5000) will be changed into $14{,}937\,c - 1998\,c^2 + 80\,c^3 - c^4$, or into $14{,}937 \times \overline{b+d} - 1998 \times \overline{b+d}^2 + 80 \times \overline{b+d}^3 - \overline{b+d}^4$, that is, into $14{,}937 \times \overline{b+d} - 1998 \times \overline{b^2 + 2bd + d^2} + 80 \times \overline{b^3 + 3b^2d + 3bd^2 + c^3} - \overline{b^4 + 4b^3d + 6b^2d^2 + 4bd^3 + d^4}$, or into the compound quantity

$$\left\{\begin{array}{rrrrrrrrr} & 14{,}937b & + & 14{,}937d & & & & & \\ - & 1998b^2 & - & 3996bd & - & 1998d^2 & & & \\ + & 80b^3 & + & 240b^2d & + & 240bd^2 & + & 80d^3 & \\ - & b^4 & - & 4b^3d & - & 6b^2d^2 & - & 4bd^3 & - d^4; \end{array}\right\}$$

which contains the quadrinomial quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, (which is equal to 5000,) together with the four quantities $14{,}937\,d$, $240\,b^2d$, $240\,bd^2$, and $80\,d^3$, which are marked with the ſign +, or are added to the quadrinomial quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and conſequently tend to increaſe it, and the ſix quantities $3996\,bd$, $1998\,d^2$, $4\,b^3d$, $6\,b^2d^2$, $4\,bd^3$, and d^4, which are marked with the ſign −, or are ſubtracted from the ſaid quadrinomial quantity, and therefore tend to diminiſh it. We muſt therefore compare the ſum of the four quantities $14{,}937\,d$, $240\,b^2d$, $240\,bd^2$, and $80\,d^3$ with the ſum of the ſix quantities $3996\,bd$, $1998\,d^2$, $4\,b^3d$, $6\,b^2d^2$,

$4bd^3$, and d^4, and inquire whether the former ſum can ever be exactly equal to the latter ſum (in which caſe $b + d$ would be another root of the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$,) or whether it muſt not be always either greater or leſs than the ſaid latter ſum, in which caſe it will be impoſſible for c, or $b + d$, or any quantity greater than b, or 13, to be a root of the ſaid equation.

Art. 77. Now, ſince b is $= 13$, we ſhall have b^2 $(= \overline{13}^2) = 169$, and b^3 $(= \overline{13}^3) = 2197$, and conſequently $240b^2d$ $(= 240 \times 169 \times d) = 40{,}560d$, and $240bd^2$ $(= 240 \times 13 \times d^2) = 3120d^2$, and $3996bd$ $(= 3996 \times 13 \times d) = 51{,}948d$, and $4b^3d$ $(= 4 \times 2197 \times d) = 8788d$, and $6b^2d^2$ $(= 6 \times 169 \times d^2) = 1014d^2$, and $4bd^3$ $(= 4 \times 13 \times d^3) = 52d^3$. Therefore the four quantities $14{,}937d + 240b^2d + 240bd^2 + 80d^3$ will be $(= 14{,}937d + 40{,}560d + 3120d^2 + 80d^3) = 55{,}497d + 3120d^2 + 80d^3$; and the ſix quantities $3996bd + 1998d^2 + 4b^3d + 6b^2d^2 + 4bd^3 + d^4$ will be $(= 51{,}948d + 1998d^2 + 8788d + 1014d^2 + 52d^3 + d^4) = 60{,}736d + 3012d^2 + 52d^3 + d^4$. We muſt therefore compare the trinomial quantity $55{,}497d + 3120d^2 + 80d^3$ with the quadrinomial quantity $60{,}736d + 3012d^2 + 52d^3 + d^4$, in order to diſcover whether it is poſſible that they ſhould be equal to each other.

Now the trinomial quantity $55{,}497d + 3120d^2 + 80d^3$ will be equal to the quadrinomial quantity $60{,}736d$

60,736 d + 3012 d^2 + 52 d^3 + d^4 if the binomial quantity 3120 d^2 + 80 d^3 is = to 60,736 d — 55,497 d + 3012 d^2 + 52 d^3 + d^4, or to 5239 d + 3012 d^2 + 52 d^3 + d^4, or if the quantity 3120 d^2 — 3012 d^2 + 80 d^3, or the binomial quantity 108 d^2 + 80 d^3 is = to the trinomial quantity 5239 d + 52 d^3 + d^4, or if the quantity 108 d^2 + 80 d^3 — 52 d^3, or the binomial quantity 108 d^2 + 28 d^3, is = the binomial quantity 5239 d + d^4, or (dividing all the terms by d) if the binomial quantity 108 d + 28 dd is equal to the binomial quantity 5239 + d^3, or if the trinomial quantity 108 d + 28 dd — d^3 is equal to the single quantity 5239.

But the trinomial quantity 108 d + 28 dd — d^3 will, in the course of the increase of x from b, or 13, *ad infinitum*, or of the increase of d (or the excess of c, or b + d, above b,) from 0 *ad infinitum* become equal to 5239; as will appear from the following considerations:

Art. 78. While d is increasing from 0 to 1, d^3 will be less than d; but when d is greater than 1, d^3 will be greater than d; and, while d increases from 1 *ad infinitum*, d^3 will become equal, first, to 108 d, and then to 108 d + 28 dd, and afterwards will be greater than 108 d + 28 dd, and will so continue, to whatever greater magnitude it may be supposed to increase. Now, when d^3 is become equal to 108 d + 28 dd, the trinomial quantity 108 d + 28 dd — d^3 will be = 0. Therefore, while d increases from 0 to the value which it has when d^3 is equal to 108 d + 28 dd, the trinomial quan-

tity $108d + 28dd - d^3$ will, firſt, increaſe from o to a certain magnitude, and then decreaſe from that magnitude to o. And conſequently, if that greateſt magnitude of the trinomial quantity $108d + 28dd - d^3$, during the ſaid increaſe of d from o to the value which it has when d^3 is equal to $108d + 28dd$, is greater than 5239, the ſaid trinomial quantity $108d + 28dd - d^3$ will, at two different inſtants of time, the one before it attains it's greateſt magnitude, and the other after it has attained it, and is decreaſing to o, be equal to 5239; and conſequently there will be two values of d which, being added to b, will make the compound quantity $14{,}937 \times \overline{b+d} - 1998 \times \overline{b+d}^2 + 80 \times \overline{b+d}^3 - \overline{b+d}^4$ be equal to $14{,}937b - 1998b^2 + 80b^3 - b^4$, or to the abſolute term 5000 of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, or there will be two values of $b + d$, or two quantities greater than b, or 13, that will be roots of that equation. Now it will be found upon examination that the greateſt magnitude which the ſaid trinomial quantity $108d + 28dd - d^3$ will attain during the ſaid increaſe of d from o to the value which it has when d^3 is equal to $108d + 28dd$, will be greater than 5239, and therefore there will be two values of $b + d$, or two quantities greater than b, or 13, that will be roots of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Theſe things may be ſhewn in the manner following:

Art. 79. In the firſt place we will inquire into the magnitude of d when d^3 is equal to $108d + 28dd$. Now

Now this value of d may be determined by reſolving a quadratick equation. For when d^3 is $= 108d + 28dd$, we ſhall (by dividing all the terms by d) have $dd = 108 + 28d$, and conſequently $dd - 28d = 108$. Therefore $dd - 28d + \overline{14}^2$ will be $(108 + \overline{14}^2 = 108 + 196) = 304$, and $d - 14$ will be $(= \sqrt{304}) = 17.4$, and d will be $(= 17.4 + 14) = 31.4$, or 31.4 will be the magnitude of d when d^3 is equal to $108d + 28dd$. Q. E. I.

Therefore, while d increaſes from 0 to 31.4, the trinomial quantity $108d + 28dd - d^3$ will, firſt, increaſe from 0 to a certain magnitude, and then will decreaſe from that magnitude and become equal to 0 again.

We will next inquire what that magnitude is, to which the ſaid trinomial quantity $108d + 28dd - d^3$ will have increaſed, and from which it will afterwards have decreaſed to 0, during the ſaid increaſe of d from 0 to 31.4, in order to diſcover whether that magnitude is greater, or leſs, than 5239. For, if it is greater than 5239, it is evident that the trinomial quantity $108d + 28dd - d^3$ will, at two different inſtants of time during the increaſe of d from 0 to 31.4, namely, once before the ſaid trinomial quantity has attained it's greateſt magnitude, and a ſecond time after it has attained it's greateſt magnitude, and while it is decreaſing from that magnitude to 0, become equal to 5239.

Now let us ſuppoſe d, or the letter d with a point placed over it, to denote ſome extremely ſmall part of

31.4, (or of the value of d when the trinomial quantity $108d + 28dd - d^3$ is become a ſecond time equal to 0,) as, for example, the 10,000,000,000th, or ten-thouſand millionth, part of it. Then it is evident that while d, in it's increaſe from 0 to 31.4, receives one of theſe ſmall increments denoted by $\dot{d}$, the ſquare of d will receive the ſmall increment $2d\dot{d}$, and d^3, or the cube of d, will receive the ſmall increment $3d^2\dot{d}$, and the quantities $108d$ and $28dd$ will receive the ſmall increments $108 \times \dot{d}$ and $28 \times 2d\dot{d}$, or $56d\dot{d}$. Therefore the increment of the binomial quantity $108d + 28dd$ will be $108\dot{d} + 56d\dot{d}$.

Now this increment $108\dot{d} + 56d\dot{d}$ will at firſt be greater than $3d^2\dot{d}$, the contemporary increment of d^3; becauſe $108 + 56d$ will at firſt be much greater than $3dd$. But, as d increaſes, the ratio of $108 + 56d$ to $3dd$ (which at firſt is a very great ratio of majority,) will for ſome time continually grow leſs and leſs and approach to a ratio of equality, and will then become a ratio of equality, and afterwards will become a ratio of minority. And when this ratio is a ratio of equality, or $108 + 56d$ is $= 3dd$, we ſhall have $\frac{108}{3} + \frac{56d}{3} = dd$, or $36 + 18.66 \times d = dd$, and conſequently $dd - 18.66 \times d = 36$, and $dd - 18.66 \times d + \overline{9.33}^2$ $(= 36 + \overline{9.33}^2 = 36 + 87.0489) = 123.0489$, and $d - 9.33$ $(= \sqrt{123.0489}) = 11.09$, and d $(= 11.09 + 9.33) = 20.42$. Now, ſo long as the increment,

ment, $108\dot{d} + 56d\dot{d}$, of the binomial quantity $108d + 28dd$ is greater than $3d^2\dot{d}$, the contemporary increment of d^3, it is evident that the trinomial quantity $108d + 28dd - d^3$, or the excefs of the binomial quantity $108d + 28dd$ above d^3, will increafe; but, when the increment of d^3 becomes greater than the increment of $108d + 28dd$, the faid trinomial quantity will decreafe. And confequently the greateft magnitude of the faid trinomial quantity $108d + 28dd - d^3$ will be that which it has when the faid increments are equal to each other, or when d is $= 20.42$.

But, when d is $= 20.42$, we fhall have $108d$ $(= 108 \times 20.42) = 2205.36$, and $28dd$ $(= 28 \times \overline{20.42}^2 = 28 \times 416.9764) = 11,675.3392$, and d^3 $(= dd \times d = 416.9764 \times 20.42) = 8514.658,088$, and confequently the trinomial quantity $108d + 28dd - d^3$ will be $(= 2205.36 + 11,675.3392 - 8514.658,088 = 13,880.6992 - 8514.658,088) = 5366.041,112$. And confequently this quantity 5366.041,112 will be the greateft magnitude of the faid trinomial quantity.

Now this quantity 5366.041,112, or (dropping the fractional part of it, as inconfiderable) 5366, is a little greater than 5239. Therefore, while d increafes from 0 to 31.4, the trinomial quantity $108d + 28dd - d^3$ will, at two different times, become equal to the quantity 5239, namely, once a little before it attains it's greateft magnitude 5366, and a fecond time a little after it has attained it's faid greateft magnitude, and is decreafing. And therefore there will be two values of d, the one a little

leſs than 20.42, and the other a little greater than 20.42, that will make the quantities $b + d$ be roots of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. So that this equation will have two roots that will be greater than 13, of which the one will be ſomewhat leſs than $b + 20.42$, or $13 + 20.42$, or 33.42, and the other will be ſomewhat greater than 33.42.

Q. E. I.

An Inquiry, whether the Equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ *may not have a fourth Root leſs than the Root* 12.756,441, *&c. already found.*

Art. 80. We will now inquire whether the ſaid equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ may not have a fourth root that ſhall be leſs than the root 12.756,441,794,480,744,022, which has been already inveſtigated, as well as the two roots that are greater than that root, and not very different from the number 33.42.

We will therefore ſuppoſe x to decreaſe from 12.756, &c, or 13, or b, to ſome leſſer quantity which we will call a, and which ſhall fall ſhort of b, or 13, by a difference called d.

Then it is evident that, when x has decreaſed from b, or 13, to a, or $b - d$, the quadrinomial quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, will be changed into the quadrinomial quantity $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times \overline{b - d}^2 + 80 \times \overline{b - d}^3 - \overline{b - d}^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times$

b^2

$\overline{b^2 - 2bd + d^2} + 80 \times \overline{b^3 - 3b^2d + 3bd^2 - d^3} -$
$\overline{b^4 - 4b^3d + 6b^2d^2 - 4bd^3 - d^4}$, or into the multinomial quantity

$$\left\{\begin{array}{l} 14{,}937\,b - 14{,}937\,d \\ -\ 1998\,b^2 + 3996\,bd - 1998\,d^2 \\ +\ 80\,b^3 - 240\,b^2d + 240\,bd^2 - 80\,d^3 \\ -\ b^4 + 4\,b^3d - 6\,b^2d^2 + 4bd^3 - d^4; \end{array}\right\}$$

which contains the quadrinomial quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, (which is equal to 5000,) together with the six quantities $14{,}937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d$, and d^4, which are marked with the sign —, or are subtracted from the quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to diminish it, and the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, which are marked with the sign +, or are added to the quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to increase it. We must therefore compare the sum of the six quantities $14{,}937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d^2$, and d^4 with the sum of the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, and inquire, whether the former sum can ever be exactly equal to the latter sum, (in which case $b - d$ would be another root of the equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$,) or whether it must not be always either greater or less than the said latter sum, in which case it will be impossible for a, or $b - d$, or any quantity less than b, or 13, to be a root of the said equation.

Art. 81.

leſs than 20.42, and the other a little greater than 20.42, that will make the quantities $b + d$ be roots of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$. So that this equation will have two roots that will be greater than 13, of which the one will be ſomewhat leſs than $b + 20.42$, or $13 + 20.42$, or 33.42, and the other will be ſomewhat greater than 33.42.

Q. E. I.

An Inquiry, whether the Equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ *may not have a fourth Root leſs than the Root* 12.756,441, *&c. already found.*

Art. 80. We will now inquire whether the ſaid equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$ may not have a fourth root that ſhall be leſs than the root 12.756,441,794,480,744,022, which has been already inveſtigated, as well as the two roots that are greater than that root, and not very different from the number 33.42.

We will therefore ſuppoſe x to decreaſe from 12.756, &c, or 13, or b, to ſome leſſer quantity which we will call a, and which ſhall fall ſhort of b, or 13, by a difference called d.

Then it is evident that, when x has decreaſed from b, or 13, to a, or $b - d$, the quadrinomial quantity $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4$, will be changed into the quadrinomial quantity $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times \overline{b - d}^2 + 80 \times \overline{b - d}^3 - \overline{b - d}^4$, or into the quadrinomial quantity $14{,}937 \times \overline{b - d} - 1998 \times$

b^2

$\overline{b^2 - 2bd + d^2} + 80 \times \overline{b^3 - 3b^2d + 3bd^2 - d^3} -$
$\overline{b^4 - 4b^3d + 6b^2d^2 - 4bd^3 - d^4}$, or into the multinomial quantity

$$\left\{\begin{array}{l} 14{,}937\,b - 14{,}937\,d \\ -\ 1998\,b^2 + 3996\,bd - 1998\,d^2 \\ +\ 80\,b^3 - 240\,b^2d + 240\,bd^2 - 80\,d^3 \\ -\ b^4 + 4\,b^3d - 6\,b^2d^2 + 4bd^3 - d^4; \end{array}\right\}$$

which contains the quadrinomial quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, (which is equal to 5000,) together with the six quantities $14{,}937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d$, and d^4, which are marked with the sign —, or are subtracted from the quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to diminish it, and the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, which are marked with the sign +, or are added to the quantity $14{,}937\,b - 1998\,b^2 + 80\,b^3 - b^4$, and consequently tend to increase it. We must therefore compare the sum of the six quantities $14{,}937\,d$, $1998\,d^2$, $240\,b^2d$, $80\,d^3$, $6\,b^2d^2$, and d^4 with the sum of the four quantities $3996\,bd$, $240\,bd^2$, $4\,b^3d$, and $4\,bd^3$, and inquire, whether the former sum can ever be exactly equal to the latter sum, (in which case $b - d$ would be another root of the equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$,) or whether it must not be always either greater or less than the said latter sum, in which case it will be impossible for a, or $b - d$, or any quantity less than b, or 13, to be a root of the said equation.

Art. 81.

Art. 81. Now, ſince b is = 13, we ſhall have $b^2 =$ 169, and $b^3 = 2197$, and $240\,b^2\,d = 40{,}560\,d$, and $240\,b\,d^2 = 3120\,d^2$, and $3996\,b\,d = 51{,}948\,d$, and $4\,b^3\,d = 8788\,d$, and $6\,b^2\,d^2 = 1014\,d^2$, and $4\,b\,d^3 = 52\,d^3$. Therefore the ſix quantities $14{,}937\,d + 1998\,d^2 + 240\,b^2\,d + 80\,d^3 + 6\,b^2\,d^2 + d^4$ will be $(= 14{,}937\,d + 1998\,d^2 + 40{,}560\,d + 80\,d^3 + 1014\,d^2 + d^4) = 55{,}497\,d + 3012\,d^2 + 80\,d^3 + d^4$; and the four quantities $3996\,b\,d + 240\,b\,d^2 + 4\,b^3\,d + 4\,b\,d^3$ will be $(= 51{,}948\,d + 3120\,d^2 + 8788\,d + 52\,d^3) = 60{,}736\,d + 3120\,d^2 + 52\,d^3$. We muſt therefore compare the quadrinomial quantity $55{,}497\,d + 3012\,d^2 + 80\,d^3 + d^4$ with the trinomial quantity $60{,}736\,d + 3120\,d^2 + 52\,d^3$, and inquire, whether it is poſſible for the former quantity to be equal to the latter.

Now the quadrinomial quantity $55{,}497\,d + 3012\,d^2 + 80\,d^3 + d^4$ will be equal to the trinomial quantity $60{,}736\,d + 3120\,d^2 + 52\,d^3$ if the trinomial quantity $3012\,d^2 + 80\,d^3 + d^4$ is equal (to the quantity $60{,}736\,d - 55{,}497\,d + 3120\,d^2 + 52\,d^3$, or) to the trinomial quantity $5239\,d + 3120\,d^2 + 52\,d^3$, or if the binomial quantity $80\,d^3 + d^4$ is equal (to the quantity $5239\,d + 3120\,d^2 - 3012\,d^2 + 52\,d^3$, or) to the trinomial quantity $5239\,d + 108\,d^2 + 52\,d^3$, or if (the quantity $80\,d^3 - 52\,d^3 + d^4$, or) the binomial quantity $28\,d^3 + d^4$ is equal to the binomial quantity $5239\,d + 108\,d^2$, or if the binomial quantity $28\,d\,d + d^3$ is equal to the binomial quantity $5239 + 108\,d$. We muſt therefore inquire whether it is poſſible for the binomial quantity

$28\,d\,d$

$28\,dd + d^3$ to be equal to the binomial quantity $5239 + 108\,d$.

Art. 82. Now d muſt in this caſe be leſs than b, becauſe it is to be ſubtracted from it. We may therefore ſuppoſe d to increaſe from 0 to b, or 12.756, or 13, but no further.

Now, when d is $= b$, or 13, we ſhall have $108\,d$ $(= 108 \times 13) = 1404$, and dd $(= b\,b = \overline{13}|^2) = 169$, and d^3 $(= b^3 = \overline{13}|^3) = 2197$, and $28\,dd$ $(= 28 \times 169) = 4732$. Therefore in this caſe the binomial quantity $28\,dd + d^3$ will be $(= 4738 + 2197) = 6929$, and the binomial quantity $5239 + 108\,d$ will be $(= 5239 + 1404) = 6643$; which is leſs than 6929. Therefore, while d increaſes from 0 to 13, or $b - d$ decreaſes from $b - 0$, or b, to $b - b$, or 0, the binomial quantity $28\,dd + d^3$ will increaſe (from $28 \times \overline{0}|^2 + \overline{0}|^3$, or) from 0 to 6929, and the binomial quantity $5239 + 108\,d$ will increaſe from $(5239 + 108 \times 0$, or $5239 + 0$, or) 5239 to (5239 + 1404, or) 6643, which is leſs than 6929, or the greateſt value of $28\,dd + d^3$. And conſequently there will be an inſtant of time during the increaſe of the binomial quantity $28\,dd + d^3$ from 0 to 6929 at which, after having been much leſs than the other binomial quantity $5239 + 108\,d$, it will become equal to it: and therefore there will be ſome value of d during it's increaſe from 0 to b, or 13, which will make the quantity $b - d$, or a, be a root of the propoſed equation $14{,}937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000$, or, in other words, the ſaid equation will have a fourth root that will be leſs than b, or 13, or 12.756,441,794,480,744,022. Q. E. I.

An

An Inveſtigation of a firſt near Value of a, *or the fourth, or leaſt, Root of the Equation* $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Art. 83. And, becauſe 6643, or the value of the binomial quantity $5239 + 108d$ when d is $= 13$, is but little leſs than 6929, or the value of the binomial quantity $28dd + d^3$ at the ſame time, it is evident that d will be but little leſs than b, or 13, when the binomial quantity $28dd + d^3$ is exactly equal to the binomial quantity $5239 + 108d$; and conſequently we may conclude that $b - d$, or a, will be but little greater than $b - b$, or 0; that is, the fourth root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = {\cdot}5000$ will be but little greater than 0, or will be only a very ſmall quantity.

Art. 84. And as a further proof that this fourth, or leaſt, value of x will be only a ſmall quantity and leſs than 1, we need only ſuppoſe x to be $= 1$, and ſubſtitute 1 inſtead of it in the terms of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$. For, if this be done, we ſhall find the ſaid compound quantity to be $(= 14{,}937 \times 1 - 1998 \times 1 \times 1 + 80 \times 1 \times 1 \times 1 - 1 \times 1 \times 1 \times 1 = 14{,}937 - 1998 + 80 - 1 = 15{,}017 - 1999) = 13{,}018$, which is almoſt triple of 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, or of the true value of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ in that equation; and conſequently 1 muſt be much greater than the ſaid true value of x, or the leaſt root of this equation. Q. E. D.

Art. 85. Since x is conſiderably leſs than 1, and probably not much greater than $\frac{1}{3}$, or 0.33, the ſeveral powers of x, to wit, x^2, x^3, and x^4, will be much leſs than x itſelf; and therefore the three terms $1998x^2$, $80x^3$, and x^4 will be much leſs than $14{,}937x$, or the firſt term of this equation; whence we may infer that, in order to obtain a tolerably near value of x, we may ſafely ſuppoſe the firſt term alone of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ to be equal to the whole four terms, and conſequently to the abſolute term 5000, and may deduce the value of x from that ſuppoſition. And then we ſhall have $x = \frac{5000}{14{,}937} = 0.33$. Therefore 0.33 will be a firſt near value of the leaſt root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, in order to diſcover whether 0.33, or this firſt near value of x is greater or leſs than it's true value, let it be ſubſtituted inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$. And we ſhall then have $x^2 (= \overline{0.33}^2) = 0.1089$, and $x^3 (= \overline{0.33}^3) = 0.035{,}937$, and $x^4 (= \overline{0.33}^4) = 0.011{,}859{,}21$, and $14{,}937x (= 14{,}937 \times 0.33) = 4929.21$ and $1998x^2 (= 1998 \times 0.1089) = 217.5822$, and $80x^3 (= 80 \times 0.035{,}937) = 2.874{,}960$. Therefore the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be $(= 4929.21 - 217.5822 + 2.874{,}960 - 0.011{,}859{,}21 = 4932.084{,}960 - 217.594{,}059{,}21) = 4714.490{,}900{,}79$, which is conſiderably leſs than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.33 will be

leſs

leſs than the true value of x, or the leaſt root of that equation. And therefore it ſeems reaſonable to conjecture that the ſaid true value will be more nearly equal to 0.35; and this ſecond near value will be very near the truth, and will be an excellent baſis for a further approach to the true value of x by Mr. Raphſon's method of approximation, which may be made in the manner following:

A further Inveſtigation of the Value of the leaſt Root of the Biquadratick Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ *by Mr. Raphſon's Method of Approximation.*

Art. 86. Since 0.35 has been found to be a near value of the leaſt root of this equation, let it be ſubſtituted inſtead of x in the terms of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the value of the ſaid compound quantity reſulting from ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the ſaid equation, and conſequently whether 0.35 will be greater, or leſs, than the true value of the leaſt root of that equation.

Now, if x is $= 0.35$, we ſhall have $x \times x$ $(= \overline{0.35}^2) = 0.1225$, and x^3 $(= \overline{0.35}^3) = 0.042{,}875$, and x^4 $(= \overline{0.35}^4) = 0.015{,}006{,}25$, and conſequently $14{,}937x$ $(= 14{,}937 \times 0.35) = 5227.95$, and $1998x^2$ $(= 1998 \times 0.1225)$

0.1225) = 244.7550, and $80x^3$ (= 80 × 0.042,875) = 3.430,000, and consequently $14{,}937x - 1998x^2 + 80x^3 - x^4$ (= 5227.95 — 244.7550 + 3.430,000 — 0.015,006,25 = 5231.380,000 — 244.770,006,25) = 4986.609,993,75; which is somewhat less than 5000, or the absolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.35 is somewhat less than the true value of x, or the least root of the said equation. Q. E. I.

Art. 87. Now let z be put for the unknown difference by which the true value of x exceeds it's near value 0.35. And we shall then have $x = 0.35 + z$, and xx (= $\overline{0.35 + z}|^2 = \overline{0.35}|^2 + 2 \times 0.35 \times z + \&c$) = $0.1225 + 0.70 \times z + \&c$,

and x^3 (= $\overline{0.35 + z}|^3 = \overline{0.35}|^3 + 3 \times \overline{0.35}|^2 \times z + \&c$
$= \overline{0.35}|^3 + 3 \times 0.1225 \times z + \&c$
$= \overline{0.35}|^3 + 0.3675 \times z + \&c)$
$= 0.042,875 + 0.3675 \times z + \&c,$

and x^4 (= $\overline{0.35 + z}|^4 = \overline{0.35}|^4 + 4 \times \overline{0.35}|^3 \times z + \&c$
$= \overline{0.35}|^4 + 4 \times 0.042,875 \times z + \&c$
$= \overline{0.35}|^4 + \quad 0.171,500 \times z + \&c)$
$= 0.015,006,25 + 0.171,500 \times z + \&c.$

Therefore $14{,}937x$ will be (= $14.937 \times \overline{0.35 + z}$ = $14{,}937 \times 0.35 + 14{,}937 \times z$) = $5227.95 + 14{,}937 \times z$; and $1998xx$ will be (= $1998 \times \overline{0.1225 + 0.70 \times z} + \&c$ = $1998 \times 0.1225 + 1998 \times 0.70 \times z + \&c$) = $244.7550 + 1{,}398.60 \times z + \&c$,

and

and $80x^3$ will be $(= 80 \times \overline{0.042,875 + 0.3675 \times z} + \&c$
$= 80 \times 0.042,875 + 80 \times 0,3675 \times z + \&c)$
$= 3.430,000 + 29.4000 \times z + \&c$; and conſequently the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{rrl} 5227.95 & + & 14,937 \times z \\ - 244.7550 & - & 1,398.60 \times z - \&c \\ + \quad 3.430,000 & + & 29.4000 \times z + \&c \\ - \quad 0.015,006,25 & - & 0.171,500 \times z - \&c, \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 5231.380,000,00 & + & 14,966.400,000 \times z + \&c \\ - 244.770,006,25 & - & 1,398.771,500 \times z - \&c \end{array}\right\}$$

$$= 4986.609,993,75 + 13,567.628,500 \times z \&c.$$

But the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the binomial quantity $4986.609,993,75 + 13,567.628,500 \times z$ will alſo be $= 5000$. And conſequently $13,567.628,500 \times z$ will be $(= 5000.000,000,00 - 4986.609,993,75) = 13.390,006,25$, and z will be $\left(= \frac{13.390,006,25}{13,567.628,500}\right) = 0.000,986,9$. Therefore x, or $0.35 + z$, will be $(= 0.35 + 0.000,986,9) = 0.350,986,9$, or, very nearly, $0.350,987$; that is, $0.350,987$ will be a third near value of x, or the leaſt root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$. Q. E. I.

Note.

Note. All the ſix figures of this value of x are exact. For, if we were to go through a ſecond proceſs of Mr. Raphſon's method of approximation, by ſuppoſing x to be equal to 0.350,987 + z, and proceeding as above, we ſhould find z to be = 0.000,000,045,866,14, and conſequently the more accurate value of x to be (0.350, 987 + 0.000,000,045,866,14, or) 0.350,987,045,866,14; of which all the figures are probably exact. But I ſhall content myſelf on the preſent occaſion with having inveſtigated this leaſt root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ to the foregoing ſix figures, 0.350,987.

Of the Two Greateſt Roots of the Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, *of which the Leſſer is ſomewhat leſs, and the greater is ſomewhat Greater, than* 32.42.

Art. 88. We will now proceed to inveſtigate to a moderate degree of exactneſs the two greateſt roots of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, which have been ſhewn above, in art. 79, to be nearly equal to 33.42, the one being a little leſs, and the other a little greater, than that number. And, firſt, we will endeavour to find the leſſer of the ſaid roots.

Now, ſince the leſſer of theſe roots is leſs than 33.42, I will, firſt, ſuppoſe it to be nearly equal to 32, and will

 ſubſtitute

ſubſtitute 32 inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the value of the ſaid compound quantity reſulting from this ſuppoſition will be greater or leſs than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, if x is $= 32$, we ſhall have $xx\ (= \overline{32}|^2) = 1024$, and $x^3\ (= \overline{32}|^3) = 32{,}768$, and $x^4\ (= \overline{32}|^4) = 1{,}048{,}576$, and $14{,}937x\ (= 14{,}937 \times 32) = 477{,}984$, and $1998xx\ (= 1998 \times 1024) = 2{,}045{,}952$, and $80x^3\ (= 80 \times 32{,}768) = 2{,}621{,}440$, and conſequently $14{,}937x - 1998x^2 + 80x^3 - x^4\ (= 477{,}984 - 2{,}045{,}952 + 2{,}621{,}440 - 1{,}048{,}576 = 3{,}099{,}424 - 3{,}094{,}528) = 4{,}896$; which is a very little leſs than 5000, or the abſolute term of the propoſed equation. Therefore 32 will be a very little leſs than x.

We will next ſuppoſe x to be $= 32.1$, and ſubſtitute 32.1 inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to try whether that value of x will make the value of the ſaid compound quantity approach nearer than the laſt value 4896 to the abſolute term 5000.

Now, if x is $= 32.1$, we ſhall have $xx\ (= \overline{32.1}|^2) = 1030.41$, and $x^3\ (= \overline{32.1}|^3) = 33{,}076.161$, and $x^4\ (= \overline{32.1}|^4) = 1{,}061{,}744.7681$, and $14{,}937x\ (= 14{,}937 \times 32.1) = 479{,}477.7$, and $1998x^2\ (= 1998 \times 1030.41) = 2{,}058{,}759.18$, and $80x^3\ (= 80 \times 33{,}076.161) = 2{,}646{,}092.880$, and conſequently

$14{,}937x$

$14{,}937x - 1998x^2 + 80x^3 - x^4 = (479{,}477.7 - 2{,}058{,}759.18 + 2{,}646{,}092.880 - 1{,}061{,}744.7681 = 3{,}125{,}570.580 - 3{,}120{,}503.9481) = 5066.6319$; which is a little greater than the abſolute term 5000 of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, but differs leſs from it than the former number 4896, ariſing from the ſubſtitution of 32, did. Therefore the true value of x will be greater than 32, but leſs than 32.1, and will probably be nearer to 32.1 than to 32.

Art. 89. We are now in poſſeſſion of three different values of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, to wit, 4896, 5000, and 5066.6319, which reſult from the ſubſtitution of the number 32, and of the true value of x, and of the number 32.1 in the ſaid compound quantity. We will now therefore, in order to obtain a more accurate value of x than either 32 or 32.1, have recourſe to the doctrine of the Scholium contained above in art. 44, 45, 46, &c - - - 54, and ſuppoſe that the exceſs of the greateſt value of x, to wit, 32.1, above the leaſt value of x, to wit, 32, will be to the exceſs of the true value of x above 32, in nearly the ſame proportion as the exceſs of the greateſt value of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, (correſponding to 32.1,) to wit, the number 5066.6319, above the leaſt value of the ſaid compound quantity, to wit, the number 4896, (correſponding to 32,) is to the exceſs of the middle value of the ſaid compound quantity, (correſponding to the true value

of x,) to wit, the abſolute term 5000 of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4$, above the leaſt value of the ſaid compound quantity, to wit, the number 4896, which correſponds to 32; that is, we will ſuppoſe $32.1 - 32$ to be to $x - 32$ in nearly the ſame proportion as $5066.6319 - 4896$ is to $5000 - 4896$, or 0.1 to be to $x - 32$ in nearly the ſame proportion as 170.6319 is to 104. And then we ſhall have $x - 32 =$, nearly, $\frac{0.1 \times 104}{170.6319}$, or (neglecting the decimal fraction 0.6319 of the denominator 170.6319,) $= \frac{0.1 \times 104}{170.} = \frac{10.4}{170} = 0.06$; and conſequently x will be $(= 0.06 + 32$, or) 32.06. Now this will be very nearly equal to the true value of x, and will therefore be an excellent baſis to a further approach to the ſaid true value by means of a proceſs of Mr. Raphſon's method of approximation, which may be performed as follows:

An Inveſtigation of a nearer Value of the Third Root of the Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ *(which is a little greater than* 32,) *by Mr. Raphſon's Method of Approximation.*

Art. 90. Let the laſt near value of x, to wit, 32.06, be ſubſtituted inſtead of x in the compound quantity $14{,}937x$

$14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or leſs than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and conſequently to determine whether 32.06 is greater, or leſs, than the true value of x.

Now, if we ſuppoſe x to be $= 32.06$, we ſhall have $xx\ (= \overline{32.06}^2) = 1027.8436$, and $x^3\ (= \overline{32.06}^3) = 32{,}952.665{,}816$, and $x^4\ (= \overline{32.06}^4) = 1{,}056{,}462.466{,}060{,}96$, and $14{,}937x\ (= 14{,}937 \times 32.06) = 478{,}880.22$, and $1998x^2\ (= 1998 \times 1027.8436) = 2{,}053{,}631.5128$, and $80x^3\ (= 80 \times 32{,}952.665{,}816) = 2{,}636{,}213.265{,}280$, and conſequently $14{,}937x - 1998x^2 + 80x^3 - x^4 = 478{,}880.22 - 2{,}053{,}631.5128 + 2{,}636{,}213.265{,}280 - 1{,}056{,}462.466{,}060{,}96 = 3{,}115{,}093.485{,}280 - 3{,}110{,}093.978{,}860{,}96 = 4999.506{,}419{,}04$; which is a very little leſs than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 32.06 muſt be a very little leſs than the true value of x in that equation.

Q. E. I.

Art. 91. Now let z be put for the unknown difference by which the true value of x, or the third root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, exceeds it's near value 32.06. And we ſhall then have $x = 32.06 + z$, and conſequently $xx\ (= \overline{32.06 + z}^2 = \overline{32.06}^2 + 2 \times 32.06 \times z + \&c = \overline{32.06}^2 + 64.12 \times z + \&c) = 1027.8436 + 64.12 \times z + \&c$,

and x^3 $(= \overline{32.06 + z}^3 \times \overline{32.06}^3 + 3 \times \overline{32.06}^2 \times z + \&c$

$= \overline{32.06}^3 + 3 \times 1027.8436 \times z + \&c$

$= \overline{32.06}^3 + \quad 3083.5308 \times z + \&c)$

$= 32,952.665,816, + 3083.5308 \times z + \&c;$

and x^4 $(= \overline{32.06 + z}^4 = \overline{32.06}^4 + 4 \times \overline{32.06}^3 \times z + \&c$

$= \overline{32.06}^4 + 4 \times 32,952.665,816 \times z + \&c$

$= \overline{32.06}^4 + \quad 131,810.663,264 \times z + \&c)$

$= 1,056,462.466,060,96 + 131,810.663,264 + z + \&c.$

Therefore $14,937x$ will be $(= 14,937 \times \overline{32.06 + z}$
$= 14,937 \times 32.06 + 14,937 \times z) = 478,880.22 + 14,937 \times z;$

and $1998x^2$ will be $(= 1998 \times \overline{1027.8436 + 64.12 \times z} + \&c$
$= 1998 \times 1027.8436 + 1998 \times 64.12 \times z + \&c)$
$= 2,053,631.5128 + 128,111.76 \times z + \&c;$

and $80\,x^3$ will be $(= 80 \times$
$\overline{32,952.665,816 + 3083.5308 \times z + \&c} = 80$
$\times 32,952.665,816 + 80 \times 3083.5308 \times z + \&c)$
$= 2,636,213.265,280 + 246,682.4640 \times z + \&c;$

and consequently the compound quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be $=$ the compound quantity

$$\left\{ \begin{array}{lll} & 478,880.22 & + 14,937 \times z \\ - & 2,053,631.5128 & - 128,111.76 \times z - \&c \end{array} \right\}$$

$$\left\{ \begin{array}{lll} + & 2,636,213.265,280 & + 246,682.4640 \times z + \&c \\ - & 1,056,462.466,060,96 & - 131,810.663,264 \times z - \&c \end{array} \right\}$$

=

$$= \left\{ \begin{array}{l} 3,115,093.485,280 \quad + 261,619.4640 \times z + \&c \\ - 3,110,093.978,860,96 - 259,922.423,264 \times z - \&c \end{array} \right\}$$

= the binomial quantity 4999.506,419,04 + 1,697.040,736 × z &c.

But the compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ is = 5000.

Therefore the binomial quantity 4999.506,419,04 + 1,697.040,736 × z will alſo be = 5000. And conſequently 1697.040,736 × z will be (= 5000.000,000,00 − 4999.506,419,04) = 0.493,580,96, and z will be = $\frac{0.493,580,96}{1697.040,736}$ = 0.000,290,848. Therefore x, or 32.06 + z, will be (= 32.06 + 0.000,290,848) = 32.060,290,848. Q. E. I.

Of this value, 32.060,290,848, of the third root of the propoſed equation $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ = 5000 I am confident that the firſt eight figures 32.060,290, are exact; and the ninth figure 8, is either exact, or too little only by an unit, the ſaid root having been found by another inveſtigation of it to be 32.060,290,901, &c.

Of the Fourth, or Greateſt, Root of the Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Art. 92. The fourth, or greateſt, root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ has been ſhewn above, in art. 79, to be ſomewhat greater than 33.42. Now, if we ſuppoſe it to be as much greater than 33.42 as the third root 32.06 is leſs than 33.42, it will be = 34.78, or nearly, = 34.8. We will therefore ſuppoſe it to be equal to 34.8, and will ſubſtitute 34.8 inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the value of that quantity reſulting from this ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and thence to determine whether 32.8 will be greater, or leſs, than the true value of the greateſt root of this equation.

Now, if we ſuppoſe x to be equal to 34.8, we ſhall have xx ($\overline{34.8}^2$) = 1211.04, and x^3 ($= \overline{34.8}^3$) = 42,144.192, and x^4 ($= \overline{34.8}^4$) = 1,466,617.8816. Therefore $14{,}937x$ will be ($= 14{,}937 \times 34.8$) = 519,807.6, and $1998x^2$ will be ($= 1998 \times 1211.04$) = 2,419,657.92, and $80x^3$ will be ($= 80 \times 42{,}144.192$) = 3,371,535.360; and conſequently the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be (=

(= 519,807.6 — 2,419,657.92 + 3,371,535.360 — 1,466,617.8816 = 3,891,342.960 — 3,886,275.8016) = 5067.1584; which is ſomewhat greater than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. And hence we may conclude that 34.8 is leſs (inſtead of being greater,) than the true value of this greateſt root of this equation; becauſe, after x, in it's increaſe from 0 *ad infinitum*, is become greater than 33.42, the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ decreaſes to 0, while x increaſes from 33.42 to the magnitude which it has when the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is = 0, or when $1998x^2 + x^4$, after having been leſs than $14{,}937x + 80x^3$, becomes equal to it, or $1998x + x^3$ becomes equal to $14{,}937 + 80x^2$, or $x^3 - 80x^2 + 1998x$ becomes equal to the known quantity 14,937, or while x increaſes from 33.42 to the quantity that is the root of the cubick equation $x^3 - 80x + 1998x = 14{,}937$. And, when x is greater than the root of this cubick equation, and increaſes from thence *ad infinitum*, the binomial quantity $x^4 + 1998x^2$ will always be greater than the binomial quantity $80x^3 + 14{,}937x$, and the quadrinomial quantity $x^4 + 1998x^2 - 80x^3 - 14{,}937x$, or $x^4 - 80x^3 + 1998x^2 - 14{,}937x$, or the exceſs of $x^4 + 1998x^2$ above $80x^3 + 14{,}937x$, will increaſe continually *ad infinitum*. But, not to dwell longer on this nice ſubject of the alternate increaſe and decreaſe of the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ during the increaſe of x from 0 to a certain magnitude, the decreaſe of this compound quantity while x increaſes from 34.8 to ſome quantity a little greater

An Inveſtigation of a Nearer Value of the Fourth, or Greateſt Root of the Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, *(which is greater than* 34.8,*) by Mr. Raphſon's Method of Approximation.*

Art. 95. Let the laſt near value of x, to wit, 34.832, be ſubſtituted inſtead of x in the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or leſs, than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and conſequently to determine whether 34.832 will be greater, or leſs, than the true value of x, or the greateſt root of that equation, which we are now inveſtigating.

Now, if we ſuppoſe x to be $= 34.832$, we ſhall have xx $(= \overline{34.832}^2) = 1213.268{,}224$,
and x^3 $(= \overline{34.832}^3) = 42{,}260.558{,}778{,}368$,
and x^4 $(= \overline{34.832}^4) = 1{,}472{,}019.783{,}368{,}114{,}176$,
and $14{,}937x$ $(= 14{,}937 \times 34.832) = 520{,}285.584$,
and $1998x^2$ $(= 1998 \times 1213.268{,}224) = 2{,}424{,}109.911{,}552$,
and $80x^3$ $(= 80 \times 42{,}260.558{,}778{,}368) = 3{,}380{,}844.702{,}269{,}440$. Therefore the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be $(= 520{,}285.584 - 2{,}424{,}109.911{,}552 + 3{,}380{,}844.702{,}269{,}440$

269,440 — 1,472,019.783,368,114,176 = 3,901,130. .286,269,440 — 3,896,129.694,920,114,176) = 5000.591,349,325,824; which is a little greater than 5000, or the abſolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

It appears therefore that, while x increaſes from 34.8 to 34.832, the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will decreaſe from 5067.1584 to 5000.591, 349,325,824, and that, while x increaſes further from 34.832 to 34.9, the ſaid compound quantity will decreaſe further from 5000.591,349,325,824 to 4852.8799. And therefore when the ſaid compound quantity is exactly equal to 5000, the value of x will be greater than 34.832, but leſs than 34.9; that is, the true value of the fourth, or greateſt, root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be ſomewhat greater than 34.832.

Art. 96. Now let z be put for the unknown difference by which the true value of x, or the fourth, or greateſt, root of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, exceeds it's near value 34.832. And we ſhall then have $x = 34.832 + z$, and conſequently

$$xx \left(= \overline{34.832 + z}^2 = \overline{34.832}^2 + z \times 34.832 \times z + \&c\right.$$
$$\left. = \overline{34.832}^2 + 69.664 \times z + \&c\right)$$
$$= 1213.268{,}224 + 69.664 \times z + \&c,$$

and $x^3 \left(= \overline{34.832 + z}^3 = \overline{34.832}^3 + 3 \times \overline{34.832}^2 \times z\right.$
$$\left. + \&c = \overline{34.832}^3 + 3 \times 1213.268{,}224 \times z + \&c\right.$$
$$=$$

$$= \overline{34.832}^3 + 3639.804,672 \times z + \&c)$$
$$= 42,260.558,778,368 + 3639.804,672 \times z + \&c,$$

and x^4 $(= \overline{34.832 + z}^4 = \overline{34.832}^4 + 4 \times \overline{34.832}^3 \times z +$
$\&c = \overline{34.832}^4 + 4 \times 42,260.558,778,368 \times z + \&c$
$= \overline{34.832}^4 + 169,042.235,113,472 \times z + \&c)$
$= 1,472,019.783,368,114,176 + 169,042.235,113,472 \times z + \&c.$

Therefore $14,937\,x$ will be $(= 14,937 \times \overline{34.832 + z}$ $= 14,937 \times 34.832 + 14,937 \times z) = 520,285.584 + 14,937 \times z$; and $1998\,x^2$ will be $(= 1998 \times \overline{1213.268,224 + 69.664 \times z + \&c} = 1998 \times 1213.268,224 + 1998 \times 69.664 \times z + \&c) = 2,424,109.911,552 + 139,188.672 \times z + \&c$; and $80\,x^3$ will be $(= 80 \times \overline{42,260.558,778,368 + 3639.804,672 \times z} = 80 \times 42,260.558,778,368 + 80 \times 3639.804,672 \times z + \&c) = 3,380,844.702,269,440 + 291,184.373,760 \times z + \&c$; and consequently the whole compound quantity $14,937\,x - 1998\,x^2 + 80\,x^3 - x^4$ will be equal to the compound quantity

$$\left\{\begin{array}{ll} 520,285.584 & + 14,937 \times z \\ -2,424,109.911,552 & - 139,188.672 \times z - \&c \\ +3,380,844.702,269,440 & + 291,184.373,760 \times z + \&c \\ -1,472,019.783,368,114,176 & - 169,042.235,113,472 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 3,901,130.286,269,440 & + 306,121.373,760 \times z + \&c \\ -3,896,129.694,920,114,176 & - 308,230.907,113,472 \times z - \&c \end{array}\right\}$$

$$= 5000.591.349,325,824 - 2,109.533,353,472 \times z \&c.$$

But

But the compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the binomial quantity $5000.591{,}349{,}325{,}824 - 2{,}109.533{,}353{,}472 \times z$ will alſo be $= 5000$; and conſequently $5000.591{,}349{,}325{,}824$ will be $= 5000 + 2{,}109.533{,}353{,}472 \times z$, and (ſubtracting 5000 from both ſides) $2{,}109.533{,}353{,}472 \times z$ will be $= 0.591{,}349{,}325{,}824$, and z will be $= \frac{0.591{,}349{,}325{,}824}{2{,}109.533{,}353{,}472} = 0.000{,}280{,}32$. Therefore x, or $34\,832 + z$ will be $(= 34.832 + 0.000{,}280{,}32) = 34.832{,}280{,}32$, or the fourth, or greateſt, root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ will be very nearly equal to $34.832{,}280{,}32$. Q. E. I.

Of this number, $34.832{,}280{,}32$, I am confident that the firſt eight figures, $34.832{,}280$, are exact, having found this root, by another inveſtigation of it, to be $= 34{,}832{,}280{,}264$, &c.

Art. 97. It appears therefore that the four roots of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ are $0.350{,}987$, &c, and $12.756{,}441{,}794{,}480{,}744{,}022$, &c, and $32.060{,}290{,}8$ &c, and $34.832{,}280$, &c. Q. E. I.

OF

OF THE DIFFERENCE BETWEEN THE ACCURATE METHODS OF RESOLVING ALGEBRÄICK EQUATIONS AND THE METHODS OF RESOLVING THEM BY APPROXIMATION.

Art. 98. An *accurate* method of reſolving an algebräick equation is a method of reſolving it, or inveſtigating the value of the unknown quantity contained in it, to any propoſed degree of exactneſs by a ſingle proceſs, or ſet of arithmetical operations of addition, ſubtraction, multiplication, diviſion, or extraction of the ſquare-root, carried to a ſufficient number of figures to attain the propoſed degree of exactneſs. And *a method of reſolving an algebräick equation by approximation* is a method of finding a *near* value of the unknown quantity contained in it by a certain proceſs of arithmetical operations which, if continued to ever ſo great a number of figures, will not approach nearer to the ſaid value than to a certain limited diſtance from it, ſo that the further continuation of the ſaid operations after a certain number of the figures of the ſaid near value have been obtained, would be intirely uſeleſs: and in this circumſtance conſiſts the inferiority of the methods of reſolving equations by approximation to the accurate methods of reſolving them. Neverthelefs, by repeating the proceſſes preſcribed by the methods of approximation we may obtain a ſecond and a third and other following near values of the unknown quantity in an equation, that ſhall approach continually

continually nearer and nearer to it's true value, till they agree with the ſaid true value to as great a degree, or to as many decimal places of figures, as we pleaſe; ſo that the repetition of the proceſſes in the methods of approximation anſwers preciſely the ſame purpoſe as the continuation of the arithmetical operations of the ſingle proceſs in the accurate methods of reſolution, by enabling us equally to attain the true value of the unknown quantity to the ſame degree of exactneſs. Of this the reſolutions of the foregoing three equations by Dr. Halley's and Mr. Raphſon's methods of approximation have afforded ſufficient proofs. And therefore the methods of reſolving equations by approximation are, in point of practical utility, as valuable as the accurate methods of reſolving them; and ſometimes they are even to be preferred to them, as, for example, in the caſe of ſuch cubick equations as come within Cardan's two rules, and in that of ſuch biquadratick equations as admit of being reſolved by the intervention of cubick equations that come within thoſe rules. For ſuch cubick equations and biquadratick equations may, for the moſt part, be more eaſily reſolved to any propoſed degree of exactneſs by two, or three, proceſſes of Mr. Raphſon's method of approximation than by the application of thoſe rules of Cardan; as has been ſhewn in ſome examples of the reſolutions of biquadratick and cubick equations in my Appendix to Mr. Frend's Principles of Algebra.

But in the reſolution of affected equations, (or equations conſiſting of more than one term involving the unknown quantity,) that are of a higher order than bi-

quadratick equations, or that involve the 5th, or 6th, or 7th, or any higher power of x, or the unknown quantity, the methods of approximation are the only methods to which we can reſort, there having been no general methods hitherto diſcovered of reſolving ſuch equations accurately, or of obtaining their roots to any propoſed degree of exactneſs by a ſingle proceſs, or courſe of arithmetical operations. And therefore theſe methods of reſolving equations by approximation muſt be allowed to be of the greateſt and moſt extenſive uſe in the buſineſs of reſolving algebräick equations, and the ingenious perſons who invented them ought to be reckoned amongſt the greateſt improvers of this branch of mathematical knowledge. Now the hiſtory of the moſt celebrated of theſe methods is briefly as follows:

A SHORT ACCOUNT OF THE PRINCIPAL METHODS OF RESOLVING ALGEBRÄICK EQUATIONS BY APPROXIMATION THAT HAVE HITHERTO BEEN PUBLISHED, AND OF THE TIMES OF THEIR PUBLICATION.

99. The oldeſt of theſe methods, as I believe, is that of the great French Algebräiſt *Vieta*, or Monſieur *Viete*, who flouriſhed about the year 1580, and died in the year 1603 at the age of 63 years, and who is deſervedly called

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The Father of Algebra, on account of the great variety and importance of his diſcoveries and improvements in that ſcience. It is contained in a pretty long diſſertation, or chapter in his Algebra, intitled, *De numeroſâ poteſtatum purarum atque adfectarum ad Exegeſin reſolutione Tractatus*, which may be ſeen in Schooten's edition of his works, publiſhed at Leyden in Holland, in the year 1646, of which it takes up 66 pages, from page 163 to page 228, incluſively. This method of reſolving equations is a very juſt and compleat one, and will enable us to find the root of any equation whatſoever to as great a degree of exactneſs as we pleaſe, by repeating the proceſſes ſufficiently often for the purpoſe, and is founded on the ſame principle as the common method of extracting the ſquare-root of a given number. But it is extreamly laborious in the application of it, on account of the number of ſubſtitutions that are to be made in every new proceſs; and it therefore requires a great deal of calculation to determine the root of the propoſed equation to any conſiderable degree of exactneſs. And for this reaſon it has ſeldom been made uſe of by mathematicians in the reſolution of equations for more than a century paſt, during which time the Publick have been in poſſeſſion of other methods of doing the ſame thing with equal certainty and exactneſs, and with much greater eaſe and expedition.

Mr. Thomas Harriot, the great Engliſh Algebraiſt, who died in the year 1621, and whoſe works were publiſhed after his death, in the year 1631, and Oughtred, who died at more than 80 years of age in the year 1660,

ſeem to have reſolved high equations by Vieta's method. And ſo did Dr. Pell and Dr. Wallis in his Algebra, (which was publiſhed in the year 1685,) and in all his former works publiſhed in the preceeding 30 years.

100. The next method of reſolving high affected equations by approximation after that of *Vieta*, was, I believe, that of Sir Iſaac Newton, which is briefly deſcribed in his moſt learned little tract, intitled *Analyſis per æquationes numero terminorum infinitas*, which he wrote about the year 1666, when he was only 24 years old, and which he communicated to Dr. Iſaac Barrow, (at that time a Fellow of Trinity-College, Cambridge, and the firſt Lucaſian Profeſſor of Mathematicks in that Univerſity,) in the year 1669. This tract of Sir Iſaac Newton was, in the ſame year 1669, communicated by Dr. Barrow, with the author's permiſſion, to Mr. John Collins; and Mr. Collins, probably, ſhewed it to many of the moſt learned mathematicians of the Royal Society. But it was not publiſhed for above forty years after, when it made part of a very valuable collection of mathematical pieces written by Sir Iſaac Newton, which was publiſhed in a thin quarto volume in the year 1711 by Mr. William Jones, an eminent mathematician in the beginning of the preſent century, and author of the mathematical work intitled, *Synopſis palmariorum Matheſeos*. And it was printed a ſecond time in the following year 1712 in the little octavo volume intitled, *Commercium Epiſtolicum de Analyſi promotâ*, which was printed by the order of the Royal Society in conſequence of the diſpute between Sir Iſaac Newton and Mr. Leibnitz concerning the

the firſt invention of the Method of Fluxions, or infinitely ſmall Differences. And it is therefore probable that many of the excellent inventions of Sir Iſaac Newton contained in that little tract intitled, *Analyſis per æquationes numero terminorum infinitas*, remained unknown to the generality of mathematicians until the year 1711. Some of them, however, had been communicated to Dr. John Wallis, Savilian Profeſſor of Geometry in the Univerſity of Oxford, ſeveral years before, and were mentioned and deſcribed by him, in a very conciſe and ſummary manner, in his Algebra, which was publiſhed in one volume folio at London in the year 1685. And amongſt theſe we find the above-mentioned method of reſolving affected equations of all degrees by approximation. See Wallis's Algebra, pages 338, 339.

101. The next method of approximating to the roots of affected equations that was made publick to the world was that of Mr. Joſeph Raphſon. This method was publiſhed in the year 1690, under the title of *Analyſis Æquationum Univerſalis, ſeu ad Æquationes Algebräicas reſolvendas Methodus generalis et expedita*, in a thin quarto volume containing about 50 pages. It is a very excellent method of reſolving all ſorts of equations by approximation in a much more eaſy and expeditious manner than that of *Vieta*, and is illuſtrated by a great number of curious and judicious examples. But it differs ſo little from that of Sir Iſaac Newton, of which Dr. Wallis had publiſhed a ſpecimen in his Algebra in the year 1685, that one can hardly avoid conjecturing that it was ſuggeſted to it's ingenious inventor by the peruſal of

that ſpecimen. But, from whatever ſource it might take it's riſe, it was a very valuable treatiſe, and was ſo fully illuſtrated by examples that it appears to have been very generally read and adopted by mathematicians, and deſervedly conſidered by them as a great improvement in Algebra. And accordingly it ſoon after became the baſis, or ground-work, of two further improvements, or ſuppoſed improvements, in this branch of Algebra, the one in France, and the other in England, to wit, the method of reſolving equations by approximation publiſhed in the Memoirs of the French Academy of Sciences in a year or two after, by *Monſieur de Lagny*, a Profeſſor of Mathematicks in the Univerſity of Paris, and that of the celebrated Mr. Edmund Halley, (afterwards Dr. Halley,) publiſhed at London in the Philoſophical Tranſactions for the year 1694, and afterwards in the year 1708 in the ſecond volume of the *Miſcellanea Curioſa*, and which is now publiſhed again at the beginning of this diſcourſe. For both theſe methods of approximating to the roots of equations ſeem manifeſtly to have been derived from Mr. Raphſon's method; and Dr. Halley confeſſes his to have been ſo. And, indeed, they are little more than variations of Mr. Raphſon's method, and ariſe ſo obviouſly from it, that one would naturally ſuppoſe that they muſt have occurred to Mr. Raphſon himſelf after he had diſcovered and perfectly underſtood his own method. And therefore it ſeems probable that, after he had carefully examined theſe methods, he did not think them, upon the whole, ſo convenient and fit to be adopted in practice as his own method. And this, (as he informs us in a ſubſequent edition of his book publiſhed in the year 1697, after

after the publication of this tract of Dr. Halley,) was really the caſe; he being of opinion, " that the greater " ſimplicity of his own method, which proceeds by the " reſolution of only ſimple equations, made it preferable " to the methods of *Monſieur de Lagny* and Dr. Halley, " which proceed either by the admiſſion of the ſquare of " the unknown quantity of the ſecond, or transformed, " equation into the expreſſion that denotes the near value " of it, after having firſt found the ſaid expreſſion for the " near value by the reſolution of a ſimple equation, or by " reſolving a quadratick equation at firſt inſtead of a " ſimple equation." And in this opinion, after a good deal of attention to the ſubject, and many trials of theſe different methods of approximating to the roots of equations, I am much inclined to agree with him; and I preſume that, after the peruſal of the reſolutions of the three equations above-mentioned both by Dr. Halley's and by Mr. Raphſon's methods of approximation, which have been given at great length in this Appendix, a majority of the readers of this tract will alſo be of the ſame opinion.

THE END.

DR. WALLIS's SOLUTION

OF

COLONEL TITUS's ARITHMETICAL PROBLEM.

An ARITHMETICAL PROBLEM, *propoſed to* Dr. WALLIS *by* Colonel SILAS TITUS.

Article 1. IN the foregoing tract of Dr. Halley on the reſolution of high affected equations by approximation, the third example adduced by him to illuſtrate his method of approximation, is the biquadratick equation $14{,}937z - 1998z^2 + 80z^3 - z^4 = 5000$, or $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$; which, he informs us, was obtained by Dr. Wallis in the 62nd chapter of his Algebra as the reſult of his ſolution of a very difficult arithmetical problem, and which Dr. Wallis has there reſolved, to a great degree of exactneſs, by the method of reſolving numeral equations, taught by Vieta in pages 163, 164, &c to page 228, of Schooten's edition of his mathematical works, publiſhed in the year 1646. See above, page 17. Now, as this equation has been ſo much the object of our attention both in Dr. Halley's own tract above-mentioned on the reſolution of equations by approximation, and in my appendix to it, (in which latter tract I have inveſtigated all it's four roots, 0.350,987,046, &c, 12.756,441,794,480,744,02 &c, 32.060,290,8 &c, and 34 832,280,2 &c,) I preſume that my readers will

An ARITHMETICAL PROBLEM, *propoſed to* Dr. WALLIS *by* Colonel SILAS TITUS.

Article 1. IN the foregoing tract of Dr. Halley on the reſolution of high affected equations by approximation, the third example adduced by him to illuſtrate his method of approximation, is the biquadratick equation $14{,}937z - 1998z^2 + 80z^3 - z^4 = 5000$, or $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$; which, he informs us, was obtained by Dr. Wallis in the 62nd chapter of his Algebra as the reſult of his ſolution of a very difficult arithmetical problem, and which Dr. Wallis has there reſolved, to a great degree of exactneſs, by the method of reſolving numeral equations, taught by Vieta in pages 163, 164, &c to page 228, of Schooten's edition of his mathematical works, publiſhed in the year 1646. See above, page 17. Now, as this equation has been ſo much the object of our attention both in Dr. Halley's own tract above-mentioned on the reſolution of equations by approximation, and in my appendix to it, (in which latter tract I have inveſtigated all it's four roots, 0.350,987,046, &c, 12.756,441,794,480,744,02 &c, 32.060,290,8 &c, and 34 832,280,2 &c,) I preſume that my readers will

not be ſorry to ſee Dr. Wallis's Solution of the Problem that gave riſe to it; more eſpecially as I ſhall endeavour to ſet forth every ſtep in the ſolution (which is very long and intricate,) in the fulleſt and cleareſt manner poſſible, ſo that the ſolution will be much more intelligible, as it will be here exhibited, than as it is exhibited in Dr. Wallis's own book, and my readers will therefore be ſpared a great deal of time and pains that I have found it neceſſary to beſtow on it before I could thoroughly underſtand it.

Art. 2. This problem was propoſed to Dr. Wallis in the year 1662, by Colonel Silas Titus, a gentleman of the bed-chamber to King Charles the Second, who was diſtinguiſhed for his knowledge of the Mathematicks, as well as of other branches of uſeful learning; and it had been originally propoſed to Colonel Titus (as the Colonel informed Dr. Wallis,) by Dr. John Pell, the famous Algebräiſt of that time. It was as follows:

AN ARITHMETICAL PROBLEM.

THERE are three numbers of ſuch magnitudes that the ſquare of the firſt number together with the product of the multiplication of the ſecond and third numbers into each other is equal to the number 16; and that the ſquare of the ſecond number together with the product of the multiplication of the firſt and third numbers into each other

other is equal to the number 17; and that the ſquare of the third number together with the product of the multiplication of the firſt and ſecond numbers into each other is equal to the number 18. It is required to find the ſaid three numbers.

THE SOLUTION.

BY DR. JOHN WALLIS,

SAVILIAN PROFESSOR OF GEOMETRY AT OXFORD, IN JUNE, 1662.

Art. 3. Let a be put for the firſt of theſe numbers, b for the ſecond, and c for the third. Then will the equation $aa + bc = 16$ expreſs the firſt condition of the problem; and the equation $bb + ac = 17$ will expreſs the ſecond condition of it; and the equation $cc + ab = 18$ will expreſs it's third condition: ſo that each of the three equations that expreſs the three conditions of the problem will involve in it all the three unknown quantities a, b, and c. And hence ariſes the great difficulty of ſolving the problem; it being neceſſary to perform a great number of laborious Algebräical operations, before we can diſentangle theſe three unknown quantities from each other, and derive from the three aforeſaid original equations $aa + bc = 16$, $bb + ac = 17$, and $cc + ab = 18$ (each of which involves all the three unknown quantities,) an equation that ſhall involve only one of them.

Art. 4.

Art. 4. The firſt thing to be ſhewn concerning theſe three unknown numbers is, "that each of them is different from the other two," or "that no two of them are equal to each other." This Dr. Wallis proves in the following manner.

In the firſt place, the firſt number a cannot be equal to the ſecond number b. For, if it were ſo, aa would be equal to bb, and bc would be $= ac$, and conſequently $aa + bc$ would be $= bb + ac$. But, by the 1ſt condition of the problem, $aa + bc$ is $= 16$; and, by the 2nd condition of the problem, $bb + ac$ is $= 17$. Therefore 16 would be $= 17$; which is impoſſible. Therefore the ſuppoſition "that a is equal to b," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Secondly, the firſt number a cannot be equal to the third number c. For, if it were ſo, aa would be $= cc$, and bc would be $= ba$, and conſequently $= ab$, and conſequently $aa + bc$ would be $= cc + ab$. But, by the 1ſt condition of the problem, $aa + bc$ is $= 16$; and, by the 3d condition of the problem, $cc + ab$ is $= 18$. Therefore 16 would be $= 18$; which is impoſſible. Therefore the ſuppoſition "that a is equal to c," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Thirdly, the ſecond number b cannot be equal to the third number c. For, if it were ſo, bb would be equal to cc, and ac would be equal to ab, and conſequently $bb + ac$ would be $= cc + ab$. But, by the ſecond condition

dition of the problem, $bb + ac$ is $= 17$; and, by the third condition of it, $cc + ab$ is $= 18$. Therefore 17 would be $= 18$; which is impoſſible. Therefore the ſuppoſition "that b is equal to c," from which this impoſſible concluſion is deduced, cannot be true. Q. E. D.

Art. 5. Having thus ſhewn that the three unknown numbers a, b, and c muſt be all different one from the other, it will be expedient, in the next place, to ſhew that c, or the third number, of which the ſquare cc occurs in the third equation $cc + ab = 18$, will be greater than either of the other two numbers a and b. This may be proved in the manner following:

Since the three unknown numbers a, b, and c are all different from each other, let the leaſt of them be denoted by the ſmall Greek letter α; and let the exceſs of the middle number above the leaſt be denoted by the ſmall Greek letter ζ, and the exceſs of the greateſt number above the middle one be denoted by the ſmall Greek letter γ. Then will the ſecond, or middle, number be $= \alpha + \zeta$, and the third, or greateſt, number will be $= \alpha + \beta + \gamma$.

Therefore the product of the multiplication of the middle number into the greateſt number will be $= \overline{\alpha + \zeta} \times \overline{\alpha + \zeta + \gamma} = \alpha^2 + \alpha\zeta + \alpha\gamma + \zeta\alpha + \zeta^2 + \zeta\gamma) = \alpha^2 + 2\alpha\zeta + \alpha\gamma + \zeta\gamma + \zeta^2$, and the ſum of the ſquare of the leaſt number and the ſaid product of the other two numbers will be $(\alpha^2 + \alpha^2 + 2\alpha\zeta + \alpha\gamma + \zeta\gamma + \zeta^2$, or) $2\alpha^2 + 2\alpha\zeta + \alpha\gamma + \zeta\gamma + \zeta^2$.

The

The ſquare of the middle number $\alpha + \mathfrak{C}$ will be $\alpha^2 + 2\alpha\mathfrak{C} + \mathfrak{C}^2$; and the product of the multiplication of the other two numbers α and $\alpha + \mathfrak{C} + \gamma$ into each other will be $\alpha^2 + \alpha\mathfrak{C} + \alpha\gamma$. Therefore the ſum of the ſaid ſquare and product will be $2\alpha^2 + 3\alpha\mathfrak{C} + \mathfrak{C}^2 + \alpha\gamma$.

The ſquare of the greateſt number $\alpha + \mathfrak{C} + \gamma$ will be $(\alpha^2 + \alpha\mathfrak{C} + \alpha\gamma + \mathfrak{C}\alpha + \mathfrak{C}^2 + \mathfrak{C}\gamma + \gamma\alpha + \gamma\mathfrak{C} + \gamma^2$, or$)$ $\alpha^2 + 2\alpha\mathfrak{C} + 2\alpha\gamma + \mathfrak{C}^2 + 2\mathfrak{C}\gamma + \gamma^2$; and the product of the multiplication of the other two numbers α and $\alpha + \mathfrak{C}$ into each other will be $\alpha^2 + \alpha\mathfrak{C}$. Therefore the ſum of the ſaid ſquare and product will be $= 2\alpha^2 + 3\alpha\mathfrak{C} + 2\alpha\gamma + \mathfrak{C}^2 + 2\mathfrak{C}\gamma + \gamma^2$.

And this laſt ſum is greater than either of the two former ſums, $2\alpha^2 + 2\alpha\mathfrak{C} + \alpha\gamma + \mathfrak{C}\gamma + \mathfrak{C}^2$ and $2\alpha^2 + 3\alpha\mathfrak{C} + \mathfrak{C}^2 + \alpha\gamma$; it's exceſs above the firſt of theſe ſums, to wit, $2\alpha^2 + 2\alpha\mathfrak{C} + \alpha\gamma + \mathfrak{C}\gamma + \mathfrak{C}^2$, being $\alpha\mathfrak{C} + \alpha\gamma + \mathfrak{C}\gamma + \gamma^2$, and it's exceſs above the ſecond ſum, to wit, $2\alpha^2 + 3\alpha\mathfrak{C} + \mathfrak{C}^2 + \alpha\gamma$, being $\alpha\gamma + 2\mathfrak{C}\gamma + \gamma^2$.

Now the leaſt of theſe three ſums muſt, by the conditions of the problem, be equal to 16, and the leaſt but one muſt be equal to 17, and the greateſt muſt be equal to 18.

Therefore, ſince the laſt ſum $2\alpha^2 + 3\alpha\mathfrak{C} + 2\alpha\gamma + \mathfrak{C}^2 + 2\mathfrak{C}\gamma + \gamma^2$, (which is $= \overline{\alpha + \mathfrak{C} + \gamma}^2 + \alpha \times \overline{\alpha + \mathfrak{C}}$,) is greater than either of the other two, it muſt be $= 18$; that is, $\overline{\alpha + \mathfrak{C} + \gamma}^2 + \alpha \times \overline{\alpha + \mathfrak{C}}$ muſt be $= 18$.

But

But $cc + ab$ is $= 18$.

Therefore $\overline{\alpha + \mathfrak{C} + \gamma}^2 + \alpha \times \overline{\alpha + \mathfrak{C}}$ muſt be $= cc + ab$; and conſequently c muſt be $= \alpha + \mathfrak{C} + \gamma$, or muſt be the greateſt of the three unknown numbers a, b, and c. Q. E. D.

Art. 6. But, whether the firſt unknown number a (of which the ſquare occurs in the firſt equation $aa + bc = 16$,) or the ſecond unknown number b, (of which the ſquare occurs in the ſecond equation $bb + ac = 17$,) is the greater, or is equal to the binomial quantity $\alpha + \mathfrak{C}$, does not yet appear. But Dr. Wallis proves that b is the greater, or is equal to $\alpha + \mathfrak{C}$, and conſequently that a is equal to α, or the leaſt of the three unknown quantities a, b, and c, by the following train of reaſoning.

Since the ſextinomial quantity $2\alpha^2 + 3\alpha\mathfrak{C} + 2\alpha\gamma + \mathfrak{C}^2 + 2\mathfrak{C}\gamma + \gamma^2$ (which is $= \overline{\alpha + \mathfrak{C} + \gamma}^2 + \alpha \times \overline{\alpha + \mathfrak{C}}$) is $= 18$, and either the quinquinomial quantity $2\alpha^2 + 2\alpha\mathfrak{C} + \alpha\gamma + \mathfrak{C}\gamma + \mathfrak{C}^2$ (which is $= \alpha^2 + \overline{\alpha + \mathfrak{C}} \times \overline{\alpha + \mathfrak{C} + \gamma}$), or the quadrinomial quantity $2\alpha^2 + 3\alpha\mathfrak{C} + \mathfrak{C}^2 + \alpha\gamma$, (which is $= \overline{\alpha + \mathfrak{C}}^2 + \alpha \times \overline{\alpha + \mathfrak{C} + \gamma}$,) is $= 17$, and the other of the ſaid two quantities is $= 16$; and the numbers 18, 17, and 16 form an arithmetical progreſſion, (the exceſs of 18 above 17, being equal to the exceſs of 17 above 16, to wit, 1);—it follows that the three quantities $2\alpha^2 + 3\alpha\mathfrak{C} + 2\alpha\gamma + \mathfrak{C}^2 + 2\mathfrak{C}\gamma + \gamma^2$,

and $2a^2 + 2a\zeta + a\gamma + \zeta\gamma + \zeta^2$ and $2a^2 + 3a\zeta + \zeta^2 + a\gamma$, (of which the firſt is equal to 18, and either the ſecond or the third is equal to 17, and either the third or the ſecond is equal to 16,) muſt alſo, if the ſecond and third were to be placed in their proper order, form an arithmetical progreſſion. Therefore, if from theſe three quantities we ſubtract the quantity $2a^2 + 2a\zeta + a\gamma + \zeta^2$, which is common to them all, the remainders after ſuch ſubtractions muſt alſo, when the ſecond and third remainders are placed in their proper order, form an arithmetical progreſſion; that is, the quadrinomial quantity, $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$, the ſingle quantity $\zeta\gamma$, and the ſingle quantity $a\zeta$, will, when $\zeta\gamma$ and $a\zeta$ are placed in their proper order, form an arithmetical progreſſion. It remains only to be determined, whether $\zeta\gamma$ is to be placed immediately after the quadrinomial quantity $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$, and before $a\zeta$, or is to be placed third in order and after $a\zeta$. Now, if $\zeta\gamma$ was to be placed ſecond, or immediately after $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$, and before $a\zeta$, it would follow from the nature of an arithmetical progreſſion (in which twice the middle term, when the number of terms is odd, is always equal to the ſum of the two terms adjoining to the middle term), that $2\zeta\gamma$ would be equal to the ſum of $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$ and $a\zeta$, that is, to $2a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$. But this is impoſſible, becauſe $2\zeta\gamma$ makes only a part of that quantity. Therefore $\zeta\gamma$ muſt not be placed next to $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$, but muſt be third in order, and muſt come after $a\zeta$; and conſequently the three terms of this arithmetical progreſſion muſt be firſt, $a\zeta + a\gamma + 2\zeta\gamma + \gamma^2$, and, ſecondly, $a\zeta$, and thirdly, $\zeta\gamma$. Therefore the former

arithmetical

arithmetical progreſſion, (from which this progreſſion was derived by ſubtracting the quadrinomial quantity $2a^2 + 2a\mathfrak{c} + \mathfrak{c}^2 + a\gamma$ from each of it's terms,) will conſiſt of the three following terms, to wit, firſt, $2a^2 + 3a\mathfrak{c} + 2a\gamma + 2\mathfrak{c}\gamma + \mathfrak{c}^2 + \gamma^2$, and, ſecondly, $2a^2 + 3a\mathfrak{c} + \mathfrak{c}^2 + a\gamma$, and thirdly, $2a^2 + 2a\mathfrak{c} + \mathfrak{c}^2 + a\gamma + \mathfrak{c}\gamma$. Therefore $2a^2 + 3a\mathfrak{c} + \mathfrak{c}^2 + a\gamma$ will be $= 17$, and $2a^2 + 2a\mathfrak{c} + \mathfrak{c}^2 + a\gamma + \mathfrak{c}\gamma$ will be $= 16$; that is, $\overline{a + \mathfrak{c}}^2 + a \times \overline{a + \mathfrak{c} + \gamma}$ will be $= 17$, and $a^2 + \overline{a + \mathfrak{c}} \times \overline{a + \mathfrak{c} + \gamma}$ will be $= 16$. Therefore $\overline{a + \mathfrak{c}}^2 + a \times \overline{a + \mathfrak{c} + \gamma}$, or $\overline{a + \mathfrak{c}}^2 + a \times c$, will be $(= 17) = bb + ac$; and conſequently b will be $= a + \mathfrak{c}$, or will be the ſecond of the three quantities a, $a + \mathfrak{c}$, and $a + \mathfrak{c} + \gamma$, or a, b, and c; and $a^2 + \overline{a + \mathfrak{c}} \times \overline{a + \mathfrak{c} + \gamma}$, or $a^2 + b \times c$, or $a^2 + bc$, will be $(= 16) = aa + bc$, and conſequently a will be $= a$, or will be the leaſt of the three unknown quantities a, $a + \mathfrak{c}$, and $a + \mathfrak{c} + \gamma$, or a, b, and c. Q. E. D.

We may therefore now conclude that a will be leſs than b, as well as that b will be leſs than c; which concluſion will be found uſeful in the courſe of the following inveſtigation of the value of a, or the firſt of the three unknown quantities a, b, and c, which is that which Dr. Wallis makes the object of his purſuit, and by means of which he afterwards determines the two other quantities b and c, and gives a compleat ſolution of the problem.

Art. 7. Now, for the ſake of brevity, let the letter l be ſubſtituted inſtead of 16 in the equation $aa + bc = 16$,

and the letter m be ſubſtituted inſtead of 17 in the equation $bb + ac = 17$, and the letter n be ſubſtituted inſtead of 18 in the equation $cc + ab = 18$. And then thoſe three original equations will be $aa + bc = l$, $bb + ac = m$, and $cc + ab = n$.

Now, ſince $aa + bc$ is $= l$, we ſhall have $bc = l - aa$, and conſequently $c = \frac{l - aa}{b}$, and $cc \left(= \frac{\overline{l - aa}^2}{bb}\right) = \frac{l^2 - 2la^2 + a^4}{bb}$.

Secondly, ſince $cc + ab$ is $= n$, we ſhall have $cc = n - ab$.

Therefore, 3dly, $n - ab$ will be $= \frac{l^2 - 2la^2 + a^4}{bb}$, and conſequently $nbb - ab^3$ will be $= l^2 - 2la^2 + a^4$.

Therefore (adding ab^3 to both ſides) nbb will be $= l^2 - 2la^2 + a^4 + ab^3$, and (adding $2la^2$ to both ſides,) $nbb + 2la^2$ will be $= l^2 + a^4 + ab^3$, and (ſubtracting $l^2 + a^4$ from both ſides,) ab^3 will be $= nbb - l^2 + 2la^2 - a^4$, and conſequently bb will be $= \frac{nbb - l^2 + 2la^2 - a^4}{ab}$.

But $bb + ac$ is $= m$, and conſequently bb is $= m - ac$.

Therefore

Therefore $\frac{nbb - l^2 + 2la^2 - a^4}{ab}$ (which is $= bb$,) will alſo be $= m - ac$.

But it has been ſhewn that c is $= \frac{l - aa}{b}$.

Therefore ac will be $(= a \times \sqrt{\frac{l-aa}{b}}) = \frac{la - a^3}{b}$, and $m - ac$ will be $= m - \sqrt{\frac{la-a^3}{b}}$ $(= \frac{mb - \sqrt{la-a^3}}{b} = \frac{mb - la + a^3}{b}) = \frac{mab - la^2 + a^4}{ab}$.

Therefore $\frac{nbb - l^2 + 2la^2 - a^4}{ab}$, (which has been ſhewn to be $m - ac$,) will be $= \frac{mab - la^2 + a^4}{ab}$; and conſequently (multiplying both ſides into ab) $nbb - l^2 + 2la^2 - a^4$ will be $= mab - la^2 + a^4$.

Therefore (adding $l^2 + a^4$ to both ſides,) we ſhall have $nbb + 2la^2 = mab - la^2 + 2a^4 + ll$, and (ſubtracting $2la^2$ from both ſides,) $nbb = mab - 3la^2 + 2a^4 + l^2$.

Now, ſince a has been ſhewn in art. 6 to be leſs than b, and m is leſs than n, (m being $= 17$, and $n = 18$) it follows that mab will be leſs than nbb, and conſequently than the quadrinomial quantity $mab - 3la^2 + 2a^4 + l^2$, which is equal to nbb. Therefore mab may be ſubtracted

from both ſides of the laſt equation $nbb = mab - 3la^2 + 2a^4 + l^2$. Let it be ſo ſubtracted. And we ſhall then have $nbb - mab = 2a^4 - 3la^2 + l^2$. Therefore (dividing both ſides by n,) we ſhall have $bb - \frac{ma}{n} \times b = \frac{2a^4 - 3la^2 + l^2}{n}$.

Art. 8. Now let $\frac{m^2a^2}{4n^2}$, or the ſquare of $\frac{ma}{2n}$, be added to both ſides of the laſt equation. And we ſhall then have $bb - \frac{ma}{n} \times b + \frac{m^2a^2}{4n^2} = \frac{2a^4 - 3la^2 + l^2}{n} + \frac{m^2a^2}{4n^2} = \frac{4n \times \overline{2a^4 - 3la^2 + l^2}}{4n \times n} + \frac{m^2a^2}{4n^2} = \frac{8na^4 - 12lna^2 + 4l^2n}{4n^2} + \frac{m^2a^2}{4n^2} = \frac{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}{4n^2}$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $b - \frac{ma}{2n} = \frac{\sqrt{8na^4 - 12.na^2 + m^2a^2 + 4l^2n}}{2n}$, and conſequently $b = \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$.

Therefore (ſquaring both ſides of this equation,) we ſhall have $bb = \frac{m^2a^2}{4n^2} + \frac{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}{4n^2} +$

$$+ \frac{2ma}{2n} \times \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} + \frac{2ma}{2n} \times$$

$$\frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} +$$

$$\frac{2ma \times \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{4n^2}.$$

But $bb + ac$ is $= m$. And ac has been shewn to be $= \frac{la - a^3}{b}$.

Therefore $bb + \frac{la - a^3}{b}$ will be $= m$.

Therefore the foregoing value of bb together with $\frac{la - a^3}{b}$ will be $= m$.

But b has been shewn to be $=$

$$\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n} =$$

$$\frac{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}.$$

Therefore $\frac{la - a^3}{b}$ will be $= \overline{la - a^3} \times$ the reciprocal of this value of b, that is, to $\overline{la - a^3} \times$ the

fraction $\frac{2n}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}$, or to the fraction $\frac{2lna - 2na^2}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}$.

Therefore the foregoing value of bb together with this laſt fraction will be $= m$; that is, the quantity

$$\frac{8na^4 - 12lna^2 + 2m^2a^2 + 4l^2n}{4n^2} + \frac{2ma \times \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{4n^2} + \frac{2lna - 2na^2}{ma + \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}$$ will be $= m$.

Art. 9. To abridge this long expreſſion, let E^2 be put $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and conſequently $E = \sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}$. And then we ſhall have $8na^4 - 12ln + 2m^2a^2 + 4l^2n$ $(= 8na^4 - 12ln + m^2a^2 + 4l^2n + m^2a^2) = E^2 + m^2a^2$; and conſequently the laſt equation will become $\frac{E^2 + m^2a^2}{4n^2} + \frac{2ma \times E}{4n^2} + \frac{2lna - 2na^2}{ma + E} = m$. Therefore (if we multiply both ſides of the equation by $4n^2$,) we ſhall have $E^2 + m^2a^2 + 2maE + \frac{8ln^2a - 8n^2a^2}{ma + E} = 4mn^2$; and (multiplying both ſides by $ma + E$) we ſhall have $maE^2 + m^3a^3 + 2m^2a^2E + E^3 + m^2a^2E + 2maE^2 + 8ln^2a - 8n^2a^2 = 4m^2n^2a + 4mn^2E$; and (ſubtracting $4m^2n^2a$ from both ſides) $maE^2 + m^3a^3 + 2m^2a^2E + E^3 + m^2a^2E$ +

$+ 2maE^2 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2E$, or $E^3 + 3maE^2 + 3m^2a^2E + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2E$; and (subtracting $E^3 + 3m^2a^2E$ from both sides) $3maE^2 + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2E - 3m^2a^2E - E^3$, and (substituting, instead of E^2 in the term $3maE^2$, it's value $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$) $24mna^5 - 36lmna^3 + 3m^3a^3 + 12l^2mna + m^3a^3 + 8ln^3a - 8n^3a^3 - 4m^2n^2a = 4mn^2E - 3m^2a^2E - E^3$, or $24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^3a^3 + 12l^2mna + 8ln^3a - 4m^2n^2a = 4mn^2E - 3m^2a^2E - E^3$. Therefore the square of the septinomial quantity $24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^3a^3 + 12l^2mna + 8ln^3a - 4m^2n^2a$ will be equal to the square of the trinomial quantity $4mn^2E - 3m^2a^2E - E^3$; that is, the compound quantity

$$
\left\{\begin{array}{llll}
576m^2n^2a^{10} & - 1728lm^2n^2a^8 & + 1872l^2m^2n^2a^6 & \\
 & + 192m^4na^8 & + 960lmn^4a^6 & \\
 & - 384mn^4a^8 & - 256m^3n^3a^6 & \\
 & & - 288lm^4na^6 & \\
 & & + 16m^6a^6 & \\
 & & + 64n^6a^6 & \\
- 864l^3m^2n^2a^4 & + 144l^4m^2n^2a^2 & & \\
- 768l^2mn^4a^4 & + 192l^3mn^4a^2 & & \\
+ 352lm^3n^3a^4 & - 96l^2n^3n^3a^2 & & \\
+ 96l^2n^4na^4 & + 64l^2n^6a^2 & & \\
- 32m^5n^2a^6 & - 64lm^3n^5a^4 & & \\
- 128ln^6a^4 & + 16m^4n^4a^2 & & \\
+ 64m^2n^5a^4 & & &
\end{array}\right\}
$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll}
 & + 72m^4na^3 & - 108lm^4na^6 & \\
 & & + 9m^6a^6 & \\
 & & - 192m^3n^3a^6 & \\
384m^2n^2a^{10} & - 1152lm^2n^2a^8 & + 1248l^2m^2n^2a^6 & \\
 & + 96m^4a^8 & - 144lm^4na^6 & \\
 & - 512mn^2a^8 & + 6m^6a^6 & \\
 & & + 1536lmn^4a^6 & \\
 & & - 128m^3n^3a^6 & \\
+ 36l^2m^4na^4 & - 96l^2m^3n^3a^6 & & \\
+ 288lm^3n^3a^4 & - 192lm^2n^5a^6 & & \\
- 24m^5n^2a^4 & + 16m^4n^4a^6 & + 64l^2m^2n^8 & \\
+ 128m^2n^5a^4 & + 96l^4m^2n^2a^6 & & \\
- 576l^3m^2n^2a^4 & + 768l^3mn^4a^6 & & \\
+ 48l^2m^4na^6 & - 64l^2m^3n^3a^6 & - 128l^4mn^6 & + z^6. \\
- 1664l^2mn^4a^6 & & & \\
+ 192lm^3n^3a^6 & & & \\
- 8m^5n^2a^4 & & &
\end{array}\right\}$$

or to the compound quantity

$$\left\{\begin{array}{llll}
384m^2n^2a^{10} & + 168m^4na^8 & - 252lm^4na^6 & \\
 & - 1152lm^2n^2a^8 & + 15m^6a^6 & \\
 & - 512mn^4a^8 & - 320m^3n^3a^6 & \\
 & & + 1248l^2m^2n^2a^6 & \\
 & & + 1536lmn^4a^6 & \\
+ 84l^2m^4na^4 & - 160l^2m^3n^3a^6 & & \\
+ 480lm^3n^3a^4 & - 192lm^2n^5a^6 & & \\
- 32m^5n^2a^4 & + 16m^4n^4a^6 & & \\
+ 128m^2n^5a^4 & + 96l^4m^2n^2a^6 & & \\
- 576l^3m^2n^2a^4 & + 768l^3mn^4a^6 & & \\
- 1664l^2mn^4a^6 & & + 64l^2m^2n^8 & \\
 & & - 128l^4mn^4 & + z^6.
\end{array}\right\}$$

Laſtly,

Laſtly, ſince E^2 is $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and E^4 is $=$ the compound quantity

$$\left\{\begin{array}{llll} 64n^2a^8 & - 192ln^2a^6 & + 208l^2n^2a^4 & - 96l^3n^2a^2 \\ + & 16m^2na^6 & - 24lm^2na^4 & + 8l^2m^2na^2 \\ & & + m^4a^4 & + 16l^4n^2, \end{array}\right\}$$

we ſhall have E^6 $=$ to the product of the multiplication of this compound quantity into the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, that is, to the compound quantity

$$\left\{\begin{array}{llll} 512n^4a^{12} & - 2304ln^3a^{10} & + 4224l^2n^3a^8 & \\ & + 192m^2n^2a^{10} & - 576lm^2n^2a^8 & \\ & & + 24m^4na^8 & \\ - 4032l^3n^3a^6 & + 2112l^4n^3a^4 & - 576l^5n^3a^2 & \\ + 624l^2m^2n^2a^6 & - 288l^3m^2n^2a^4 & + 48l^4m^2n^2a^2 & \\ - 36lm^4na^6 & + 12l^2m^4na^4 & & \\ + m^6a^6 & & + 64l^6n^3. & \end{array}\right\}$$

Therefore the compound quantity

$$\left\{\begin{array}{lll} 9m^4a^4E^2 & + 6m^2a^2E^4 & \\ - 24m^3n^2a^2E^2 & - 8mn^2E^4 & + E^6 \\ + 16m^2n^4E^2 & & \end{array}\right\}$$

will be equal to the following very numerous compound quantity, to wit,

$512n^4a^{12}$

$$
\left\{\begin{array}{llll}
 & + 72m^4na^8 & - 108lm^4na^6 & \\
 & & + 9m^6a^6 & \\
 & & - 192m^3n^3a^6 & \\
384m^2n^2a^{10} & - 1152lm^2n^2a^8 & + 1248l^2m^2n^2a^6 & \\
 & + 96m^4na^8 & - 144lm^4na^6 & \\
 & - 512mn^4a^8 & + 6m^6a^6 & \\
 & & + 1536lmn^4a^6 & \\
 & & - 128m^3n^3a^6 & \\
+ 36l^2m^4na^4 & - 96l^2m^3n^3a^6 & & \\
+ 288lm^3n^3a^4 & - 192lm^2n^5a^6 & & \\
- 24m^5n^2a^4 & + 16m^4n^4a^6 & + 64l^2m^2n^5 & \\
+ 128m^2n^5a^4 & + 96l^4m^2n^2a^6 & & \\
- 576l^3m^2n^2a^4 & + 768l^3mn^4a^6 & & \\
+ 48l^2m^4na^6 & - 64l^2m^3n^3a^6 & - 128l^4mn^6 & + z^6. \\
- 1664l^2mn^4a^6 & & & \\
+ 192lm^3n^3a^4 & & & \\
- 8m^5n^2a^4 & & &
\end{array}\right.
$$

or to the compound quantity

$$
\left\{\begin{array}{llll}
384m^2n^2a^{10} & + 168m^4na^8 & - 252lm^4na^6 & \\
 & - 1152lm^2n^2a^8 & + 15m^6a^6 & \\
 & - 512mn^4a^8 & - 320m^3n^3a^6 & \\
 & & + 1248l^2m^2n^2a^6 & \\
 & & + 1536lmn^4a^6 & \\
+ 84l^2m^4na^4 & - 160l^2m^3n^3a^6 & & \\
+ 480lm^3n^3a^4 & - 192lm^2n^5a^6 & & \\
- 32m^5n^2a^4 & + 16m^4n^4a^6 & & \\
+ 128m^2n^5a^4 & + 96l^4m^2n^2a^6 & & \\
- 576l^3m^2n^2a^4 & + 768l^3mn^4a^6 & & \\
- 1664l^2mn^4a^6 & & + 64l^2m^2n^5 & \\
 & & - 128l^4mn^4 & + z^6.
\end{array}\right.
$$

Laſtly,

Laſtly, ſince E^2 is $= 8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, and E^4 is $=$ the compound quantity

$$\left\{\begin{array}{llll} 64n^2a^8 & - 192ln^2a^6 & + 208l^2n^2a^4 & - 96l^3n^2a^2 \\ + & 16m^2na^6 & - 24lm^2na^4 & + 8l^2m^2na^2 \\ & & + m^4a^4 & + 16l^4n^2, \end{array}\right\}$$

we ſhall have $E^6 =$ to the product of the multiplication of this compound quantity into the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$, that is, to the compound quantity

$$\left\{\begin{array}{llll} 512n^4a^{12} & - 2304ln^3a^{10} & + 4224l^2n^3a^8 & \\ + & 192m^2n^2a^{10} & - 576lm^2n^2a^8 & \\ & & + 24m^4na^8 & \\ - 4032l^3n^3a^6 & + 2112l^4n^3a^4 & - 576l^5n^3a^2 & \\ + 624l^2m^2n^2a^6 & - 288l^3m^2n^2a^4 & + 48l^4m^2n^2a^2 & \\ - 36lm^4na^6 & + 12l^2m^4na^4 & & \\ + m^6a^6 & & + 64l^6n^3. & \end{array}\right\}$$

Therefore the compound quantity

$$\left\{\begin{array}{lll} 9m^4a^4E^2 & + 6m^2a^2E^4 & \\ - 24m^3n^2a^2E^2 & - 8mn^2E^4 & + E^6 \\ + 16m^2n^4E^2 & & \end{array}\right\}$$

will be equal to the following very numerous compound quantity, to wit,

$512n^5a^{12}$

$$
\begin{array}{lll}
512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 \\
 & +192m^2n^2a^{10} & -576lm^2n^2a^8 \\
 & & +24m^4na^8 \\
 & +384m^2n^2a^{10} & +168m^4na^8 \\
 & & -1152lm^2n^2a^8 \\
 & & -512mn^4a^8
\end{array}
$$

$$
\begin{array}{lll}
-4032l^3n^3a^6 & +2112l^4n^3a^6 & -576l^5n^3a^2 \\
+624l^3m^2n^2a^6 & -288l^3m^2n^2a^6 & +48l^4m^2n^2a^2 \\
-36lm^4na^6 & +12l^2m^4na^4 & \\
+m^6a^6 & & \\
-252lm^4na^6 & +84l^2m^4na^4 & -160l^2m^3n^3a^2 \\
+15m^6a^6 & +480lm^3n^3a^4 & -192lm^2n^5a^2 \\
-320m^3n^3a^6 & -32m^5n^2a^4 & +16m^4n^4a^2 \\
 & +128m^2n^5a^4 & \\
+1248l^2m^2n^2a^6 & -576l^3m^2n^2a^4 & +96l^4m^2n^2a^2 \\
+1536lmn^4a^6 & -1664l^2mn^4a^4 & +768l^3mn^4a^2 \\
 & & +64l^6n^3 \\
 & & +64l^3m^2n^4 \\
 & & -128l^4mn^4
\end{array}
$$

or to the compound quantity

$$
\begin{array}{llll}
512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 & -4032l^3n^3a^6 \\
 & +576m^2n^2a^{10} & -1728lm^2n^2a^8 & +1872l^2m^2n^2a^6 \\
 & & +192m^4na^8 & -288lm^4na^6 \\
 & & -512mn^4a^8 & +16m^6a^6 \\
 & & & -320m^3n^2a^6 \\
 & & & +1536lmn^4a^6
\end{array}
$$

$$
\begin{array}{lll}
+2112l^4n^3a^4 & -576l^5n^3a^2 & +64l^6n^3 \\
-864l^3m^2n^2a^4 & +144l^4m^2n^2a^2 & +64l^3m^2n^4 \\
+96l^2m^4na^4 & -160l^2m^3n^3a^2 & -128l^4mn^4 \\
+480lm^3n^3a^4 & -192lm^2n^5a^2 & \\
-32m^5n^2a^4 & +16m^4n^4a^2 & \\
+128m^2n^5a^4 & +768l^3mn^4a^2. & \\
-1664l^2mn^4a^4 & &
\end{array}
$$

Therefore

Therefore the compound quantity which forms the firſt, or left-hand, ſide of the equation ſet down at the end of art. 9, to wit, the compound quantity

$$\left\{\begin{array}{llll} 576m^2n^2a^{10} & - 1728lm^2n^2a^8 & + 1872l^2m^2n^2a^6 & \\ & + 192m^4na^8 & + 960lmn^4a^6 & \\ & - 384mn^4a^8 & - 256m^3n^3a^6 & \\ & & - 288lm^4na^6 & \\ & & + 16m^6a^6 & \\ & & + 64n^6a^6 & \\ - 864l^3m^2n^2a^4 & + 144l^4m^2n^2a^2 & & \\ - 768l^2mn^4a^4 & + 192l^3mn^4a^2 & & \\ + 352lm^3n^3a^4 & - 96l^2m^3n^3a^2 & & \\ + 96l^2m^4na^4 & + 64l^2n^6a^2 & & \\ - 32m^5n^2a^4 & - 64lm^2n^5a^2 & & \\ - 128ln^6a^4 & + 16m^4n^4a^2 & & \\ + 64m^2n^5a^4 & & & \end{array}\right\}$$

(which is equal to the compound quantity

$$\left\{\begin{array}{lll} 9m^4a^4E^2 & + 6m^2a^2E^4 & \\ - 24m^3n^2a^2E^2 & - 8mn^2E^4 & + E^6) \text{ will be equal} \\ + 16m^2n^4E^2 & & \end{array}\right.$$

to the compound quantity

$$512n^3a^{12}$$

$$
\begin{array}{llll}
512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 & \\
 & +576m^2n^2a^{10} & -1728lm^2n^2a^8 & \\
 & & +192m^4na^8 & \\
 & & -512mn^4a^8 & \\
-4032l^3n^3a^6 & & +2112l^4n^3a^4 & \\
+1872l^2m^2n^2a^6 & & -864l^3m^2n^2a^4 & \\
-288lm^4na^6 & & +96l^2m^4na^4 & \\
+16m^6a^6 & & +480lm^3n^3a^4 & \\
-320m^3n^3a^6 & & -32m^5n^2a^4 & \\
+1536lmn^4a^6 & & +128m^3n^3a^4 & \\
 & & -1664l^2mn^4a^4 & \\
 & -576l^3n^3a^2 & & \\
 & +144l^4m^2n^2a^2 & & \\
 & -160l^2m^3n^3a^2 & & \\
 & -192lm^2n^3a^2 & +64l^3n^3 & \\
 & +16m^4n^4a^2 & +64l^2m^2n^3 & \\
 & +768l^3mn^4a^2 & -128l^4mn^4. &
\end{array}
$$

Art. 11. Now the following eleven terms occur in both theſe compound quantities, and with the ſame ſigns + and — prefixed to them, to wit, the terms

$+576m^2n^2a^{10}$, $-1728lm^2n^2a^8$, $+1872l^2m^2n^2a^6$,
$+192m^4na^8$, $-288lm^4na^6$,
$+16m^6a^6$,
$-864l^3m^2n^2a^4$, $+144l^4m^2n^2a^2$,
$+96l^2m^4na^4$, $+16m^4n^4a^2$. Therefore, if theſe
$-32m^5n^2a^4$,

terms are omitted, the remaining terms of theſe two compound quantities will ſtill be equal to each other; that is, the compound quantity

—

$$\left\{\begin{array}{llll} -384mn^4a^8 & +960l^1mn^4a^6 & -768l^2mn^4a^4 \\ & -256m^3n^3a^6 & +352lm^3n^3a^4 \\ & +64n^6a^6 & -128l^1n^6a^4 \\ & & +64m^2n^5a^4 \\ & +192l^3mn^4a^2 & \\ & -96l^2m^3n^3a^2 & \\ & +64l^2n^6a^2 & \\ & -64lm^2n^5a^2 & \end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll} 512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 & \\ & & -512mn^4a^8 & \\ -4032l^3n^3a^6 & +2112l^4n^3a^4 & -576l^5n^3a^2 & \\ -320m^3n^3a^6 & +480lm^3n^3a^4 & -160l^2m^3n^3a^2 & \\ +1536lmn^4a^6 & +128m^2n^5a^4 & -192lm^2n^5a^2 & \\ & -1664l^2mn^4a^4 & +768l^3mn^4a^2 & \\ & & +64l^6n^8 & \\ & & +64l^2m^2n^5 & \\ & & -128l^4mn^4. & \end{array}\right\}$$

Add $512mn^4a^8 + 320m^3n^3a^6 + 1664l^2mn^4a^4 + 160l^2m^3n^3a^2 + 192lm^2n^5a^2$ to both ſides. And we ſhall then have the compound quantity

$$\left\{\begin{array}{llll} 128mn^4a^9 & +\ 960l^1mn^4a^6 & +\ 896l^2mn^4a^4 & \\ & +\ 64m^3n^3a^6 & +\ 352lm^3n^3a^4 & \\ & +\ 64n^6a^6 & -\ 128l^3a^4 & \\ & & +\ 64m^2n^5a^4 & \\ +\ 192l^2mn^4a^2 & & & \\ +\ 64l^2m^3n^3a^2 & & & \\ +\ 64l^2n^6a^2 & & & \\ +\ 128lm^2n^5a^2 & & & \end{array}\right\}$$

= the compound quantity

$$\left\{\begin{array}{llll} 512n^3a^{12} & -\ 2304ln^3a^{10} & +\ 4224l^2n^3a^8 & -\ 4032l^3n^3a^6 \\ & & & +\ 1536lmn^4a^6 \\ +\ 2112l^4n^3a^4 & & -\ 576l^5n^3a^2 & \\ +\ 480lm^3n^3a^4 & +\ 768l^3mn^4a^2 & +\ 64l^6n^3 & \\ +\ 128m^2n^5a^4 & & +\ 64l^2m^2n^5 & \\ & & -\ 128l^4mn^4. & \end{array}\right\}$$

Now let $960lmn^4a^6 + 352lm^3n^3a^4 + 64m^2n^5a^4 + 192l^2mn^4a^2$ be subtracted from both sides. And we shall then have the compound quantity

$$\left\{\begin{array}{llll} 128mn^4a^8 & +\ 64m^3n^3a^6 & +\ 896l^2mn^4a^4 & +\ 64l^2m^3n^3a^2 \\ & +\ 64n^6a^6 & -\ 128l^3a^4 & +\ 64l^2n^6a^2 \\ & & & +\ 128lm^2n^5a^2 \end{array}\right\}$$

= the compound quantity

$512n^3a^{12}$

$$\left\{\begin{array}{llll} 512n^3a^{12} & -2304ln^3a^{10} & +4224l^2n^3a^8 & \\ & -4032l^3n^3a^6 & +2112l^4n^3a^4 & \\ & +576lmn^4a^6 & +128lm^3n^3a^4 & \\ & & +64m^2n^5a^4 & \\ & -576l^5n^3a^2 & & \\ & +576l^3mn^4a^2 & +64l^6n^3 & \\ & & +64l^2m^2n^5 & \\ & & -128l^4mn^4; & \end{array}\right\}$$

and (dividing all the terms by n^3,) the compound quantity

$$\left\{\begin{array}{llll} 128mna^8 & +64m^3a^6 & +896l^2mna^4 & \\ & +64n^3a^6 & -128ln^3a^4 & \\ & & +64l^2m^3a^2 & \\ & & +64l^2n^3a^2 & \\ & & +128lm^2n^2a^2 & \end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll} 512a^{12} & -2304la^{10} & +4224l^2a^8 & -4032l^3a^6 \\ & & & +576lmna^6 \\ & +2112l^4a^4 & -576l^5a^2 & \\ & +128lm^3a^4 & +576l^3mna^2 & +64l^6 \\ & +64m^2n^2a^4 & & +64l^2m^2n^2 \\ & & & -128l^4mn; \end{array}\right\}$$

and (dividing all the terms by 32,) the compound quantity

$$\left\{\begin{array}{llll} 4mna^8 & +2m^2a^6 & +28l^2mna^4 & \\ & +2n^2a^6 & -4ln^3a^4 & \\ & & +2l^2m^3a^2 & \\ & & +2l^2n^3a^2 & \\ & & +4l^4mna^2 & \end{array}\right\}$$

will be equal to the compound quantity

$$\left\{\begin{array}{llll} 16a^{12} & -72la^{10} & +132l^2a^8 & -126l^3a^6 \\ & & & +18lmna^6 \\ & +66l^4a^4 & -18l^5a^2 & +2l^6 \\ & +4lm^3a^4 & +18l^3mna^2 & +2l^2m^2n^2 \\ & +2m^2n^2a^4 & & -4l^4mn \end{array}\right\};$$

and (fubtracting the former of theſe compound quantities from the latter,) we ſhall have the compound quantity

$$\left\{\begin{array}{llll} 16a^{12} & -72la^{10} & +132l^2a^8 & -126l^3a^6 \\ & & & +18lmna^6 \\ & & -4mna^8 & -2m^2a^6 \\ & & & -2n^3a^6 \\ & +66l^4a^4 & & \\ & +4lm^3a^4 & -18l^5a^2 & \\ & +2m^2n^2a^4 & +18l^3mna^2 & +2l^6 \\ & -28l^2mna^4 & -l^3m^3a^2 & +2l^2m^2n^2 \\ & +4ln^3a^4 & -2l^2n^3a^2 & -4l^4mn \\ & & -4lm^2n^2a^2 & \end{array}\right\}$$

$$= C.$$

Art. 12.

Art. 12. Now let all the terms of this equation be divided by 16, in order to free the higheſt power of the unknown quantity a from it's co-efficient. And we ſhall then have

$$\begin{aligned}
a^{12} &- \tfrac{9}{2} la^{10} + \tfrac{33}{4} l^2a^8 - \tfrac{63}{8} l^3a^6 + \tfrac{33}{8} l^4a^4 \\
&- \tfrac{1}{4} mna^8 + \tfrac{9}{8} lmna^6 + \tfrac{1}{4} lm^3a^4 \\
&- \tfrac{1}{8} m^3a^6 + \tfrac{1}{8} m^2n^2a^4 \\
&- \tfrac{1}{8} n^3a^6 - \tfrac{7}{4} l^2mna^4 \\
&+ \tfrac{1}{4} ln^3a^4 \\
&- \tfrac{9}{8} l^3a^2 + \tfrac{1}{8} l^6 \\
&+ \tfrac{9}{8} l^3mna^2 + \tfrac{1}{8} l^2m^2n^2 \\
&- \tfrac{1}{8} l^2m^3a^2 - \tfrac{1}{4} l^4mn \\
&- \tfrac{1}{8} l^2n^3a^2 \\
&- \tfrac{1}{4} lm^2n^2a^2 \\
&= 0.
\end{aligned}$$

This laſt equation may be free'd from fractions by taking $ee = 2aa$, or $\frac{ee}{2} = aa$, and ſubſtituting $\frac{ee}{2}$

inſtead of aa in it's terms. For, if $\frac{ee}{2}$ is $= aa$, we ſhall have $\frac{e^4}{4} = a^4$, and $\frac{e^6}{8} = a^6$, and $\frac{e^8}{16} = a^8$, and $\frac{e^{10}}{32} = a^{10}$, and $\frac{e^{12}}{64} = a^{12}$. And conſequently, by this ſubſtitution of $\frac{ee}{2}$ inſtead of aa, the laſt equation will be converted into the following equation, to wit,

$$\left.\begin{array}{rrrrr}
\frac{e^{12}}{64} - \frac{9}{2} \times l \times \frac{e^{10}}{32} + \frac{33}{4} \times l^2 \times \frac{e^8}{16} - \frac{63}{8} \times l^3 \times \frac{e^6}{8} & & & & \\
- \frac{1}{4} \times mn \times \frac{e^8}{16} + \frac{9}{8} \times lmn \times \frac{e^6}{8} & & & & \\
- \frac{1}{8} \times m^3 \times \frac{e^6}{8} & & & & \\
- \frac{1}{8} n^3 \times \frac{e^6}{8} & & & & \\
+ \frac{33}{8} \times l^4 \times \frac{e^4}{4} & - \frac{9}{8} \times l^5 \times \frac{e^2}{2} & & & \\
+ \frac{1}{4} \times lm^2 \times \frac{e^4}{4} & + \frac{9}{8} \times l^3 mn \times \frac{e^2}{2} & + \frac{1}{8} l^6 & & \\
+ \frac{1}{8} \times m^2 n^2 \times \frac{e^4}{4} & - \frac{1}{8} \times l^2 m^3 \times \frac{e^2}{2} & + \frac{1}{8} l^2 m^2 n^2 & & \\
- \frac{7}{4} \times l^2 mn \times \frac{e^4}{4} & - \frac{1}{8} \times l^2 n^3 \times \frac{e^2}{2} & - \frac{1}{4} l^4 mn & & \\
+ \frac{1}{4} \times ln^3 \times \frac{e^4}{4} & - \frac{1}{4} \times lm^2 n^2 \times \frac{e^2}{2} & & &
\end{array}\right\} = 0, \text{ or}$$

$$\frac{e^{12}}{64}$$

$$\left\{\begin{array}{llllll}
\frac{e^{12}}{64} & -\frac{9le^{10}}{64} & +\frac{33l^2e^9}{64} & -\frac{63l^3e^8}{64} & +\frac{33l^4e^7}{32} & \\
 & -\frac{mne^8}{64} & +\frac{9lmne^6}{64} & +\frac{lm^3e^4}{16} & & \\
 & & -\frac{m^3e^6}{64} & +\frac{m^2n^2e^4}{32} & & \\
 & & -\frac{n^2e^6}{64} & -\frac{7l^2mne^4}{16} & & \\
 & & & +\frac{ln^3e^4}{16} & & \\
 & -\frac{9l^5e^2}{16} & & & & \\
 & +\frac{9l^3mne^2}{16} & +\frac{1}{8}l^5 & & & \\
 & -\frac{l^2m^3e^2}{16} & +\frac{1}{8}l^3m^2n^2 & & & \\
 & -\frac{l^2n^3e^2}{16} & -\frac{1}{4}l^4mn & & & \\
 & -\frac{lm^2n^2e^2}{8} & & & &
\end{array}\right\}$$

$$= 0, \text{ or}$$

$$\frac{e^{12}}{64}$$

$$\left\{\begin{array}{llllll}
\frac{e^{12}}{64} & -\frac{9le^{10}}{64} & +\frac{33l^2e^8}{64} & -\frac{63l^3e^6}{64} & +\frac{66l^4e^4}{64} & \\
 & & -\frac{mne^8}{64} & +\frac{9lmne^6}{64} & +\frac{4lm^3e^4}{64} & \\
 & & & -\frac{m^3e^6}{64} & +\frac{2m^2n^2e^4}{64} & \\
 & & & -\frac{n^3e^6}{64} & -\frac{28l^2mne^4}{64} & \\
 & & & & +\frac{4ln^3e^4}{64} & \\
 & & -\frac{36l^5e^2}{64} & +\frac{8l^6}{64} & & \\
 & & +\frac{36l^3mne^2}{64} & +\frac{8l^2m^2n^2}{64} & & \\
 & & -\frac{4l^2m^3e^2}{64} & -\frac{16l^4mn}{64} & & \\
 & & -\frac{4l^2n^3e^2}{64} & & & \\
 & & -\frac{8lm^2n^2e^2}{64} & & &
\end{array}\right\}$$

$= 0$; and consequently (multiplying all the terms by 64) we ſhall have

e^{12}

$$\left\{\begin{array}{llllll} e^{12} & -9le^{10} & +33l^2e^8 & -63l^3e^6 & +66l^4e^4 & \\ & -mne^8 & +9lmne^6 & +4lm^3e^4 & & \\ & & -m^3e^6 & +2m^2n^2e^4 & & \\ & & -n^3e^6 & -28l^2mne^4 & & \\ & & & +4ln^3e^4 & & \\ -36l^5e^2 & & +8l^6 & & & \\ +36l^3mne^2 & & +8l^2m^2n^2 & & & \\ -4l^2m^3e^2 & & -16l^4mn & & & \\ -4l^2n^3e^2 & & & & & \\ -8lm^2n^2e^2 & & & & & \end{array}\right\}$$

$$= 0.$$

A Reduction of the foregoing Equation of the Twelfth Order to an Equation of the Eighth Order by dividing it by the Trinomial Quantity $e^4 - 4le^2 + 4l^2$.

Art. 13. This equation may be reduced to an equation of the eighth power by dividing it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, or the ſquare of $2l - e^2$; which Diviſion is ſet down by Dr. Wallis in his Algebra, chapter 61, and is as follows:

The

Therefore the quotient arising from the division of the foregoing dividend, (which was the first, or left-hand, side of the last equation, and was equal to 0,) is the compound quantity

$$\left\{\begin{array}{llll} e^8 - 5le^6 & + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^2e^2 & + 2m^2n^2. \\ & & - n^2e^2 & \end{array}\right\}$$

And consequently, as the said dividend was equal to 0, this last compound quantity, which arises from the division of it by the trinomial quantity $e^4 - 4le^2 + 4l^2$, must be equal to 0 likewise; and therefore the final equation, by the resolution of which the value of ee, and consequently those of aa and a in Colonel Titus's Problem, is to be obtained, will be the equation

$$\left\{\begin{array}{llll} e^8 - 5le^6 & + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 & + 5lmne^2 & - 4l^2mn \\ & & - m^2e^2 & + 2m^2n^2 \\ & & - n^2e^2 & \end{array}\right\} = 0.$$

Art. 14. Now let the values of the three letters l, m, and n, be substituted, instead of those letters themselves, in the terms of this equation, to wit, 16 instead of l, 17 instead of m, and 18 instead of n. And then we shall have $5le^6$ $(= 5 \times 16 \times e^6) = 80e^6$; and $9l^2e^4$ $(= 9 \times 16 \times 16 \times e^4 = 9 \times 256 \times e^4) = 2304e^4$, and mne^4 $(= 17 \times 18 \times e^4) = 306e^4$, and consequently $9l^2e^4 - mne^4$ $(= 2304e^4 - 306e^4) = 1998e^4$; and

$7l^3e^2$

$7l^3e^2$ ($= 7 \times 16 \times 16 \times 16 \times e^2 = 7 \times 256 \times 16 \times e^2 = 7 \times 4096 \times e^2$) $= 28{,}672e^2$, and $5lmne^2$ ($= 5 \times 16 \times 17 \times 18 \times e^2 = 80 \times 17 \times 18 \times e^2 = 80 \times 306 \times e^2$) $= 24{,}480e^2$, and m^3e^2 ($= 17 \times 17 \times 17 \times e^2 = 289 \times 17 \times e^2$) $= 4913e^2$, and n^3e^2 ($= 18 \times 18 \times 18 \times e^2 = 324 \times 18 \times e^2$) $= 5832e^2$, and $7l^3e^2 + m^3e^2 + n^3e^2$ ($= 28{,}672e^2 + 4913e^2 + 5832e^2$) $= 39{,}417e^2$, and consequently $7l^3e^2 + m^3e^2 + n^3e^2 - 5lmne^2$ ($= 39{,}417e^2 - 24{,}480e^2 = 14{,}937e^2$; and, lastly, $2l^4$ ($= 2 \times l^3 \times l = 2 \times 4096 \times 16 = 2 \times 65{,}536$) $= 131{,}072$, and $2m^2n^2$ ($= 2 \times 17 \times 17 \times 18 \times 18 = 2 \times 289 \times 324 = 2 \times 93{,}636$) $= 187{,}272$, and $2l^4 + 2m^2n^2$ ($= 131{,}072 + 187{,}272$) $= 318{,}344$, and $4l^2mn$ ($= 4 \times 256 \times 17 \times 18 = 4 \times 256 \times 306 = 4 \times 78{,}336$) $= 313{,}344$, and consequently $2l^4 + 2m^2n^2 - 4l^2mn$ ($= 318{,}344 - 313{,}344$) $= 5000$. Therefore the equation

$$\left\{\begin{array}{llll} e^8 - 5le^6 + 9l^2e^4 & - 7l^3e^2 & + 2l^4 \\ & - mne^4 + 5lmne^2 & - 4l^2mn \\ & - m^3e^2 & + 2m^2n^2 \\ & - n^3e^2 \end{array}\right\}$$

$= 0$, will, when the values of l, m, n are ſubſtituted inſtead of thoſe letters in the ſeveral terms of the equation, and the ſeveral neceſſary additions and ſubtractions of the terms involving the ſame powers of ee have been duly made, become $e^8 - 80e^6 + 1998e^4 - 14{,}937e^2 + 5000 = 0$; and therefore (adding $80e^6 + 14{,}937e^2$ to both ſides) we ſhall have $e^8 + 1998e^4 + 5000 = 14{,}937e^2$

$14{,}937e^2 + 80e^6$; and (ſubtracting $e^8 + 1998e^4$ from both ſides,) we ſhall have $5000 = 14{,}937e^2 + 80e^6 - e^8 - 1998e^4$, or $14{,}937e^2 - 1998e^4 + 80e^6 - e^8 = 5000$. And laſtly, ſubſtituting x inſtead of ee, we ſhall have the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, which has been reſolved by Dr. Halley in the foregoing tract, and more fully in the Appendix to the ſaid tract, in which all it's four roots have been inveſtigated, and have been ſhewn to be 0.350,987,046, and 12.756,441,794,480,744,02, &c, and 32.060,290,8, and 34.832,280,2.

Another Method of reducing the foregoing Equation of the Twelfth Order to an Equation of the Eighth Order by means of two Diviſions by the ſame Diviſor as before.

Art. 15. The foregoing reduction of the equation of the twelfth power to an equation of the eighth power by means of the diviſion of it by the trinomial quantity $e^4 - 4le^2 + 4l$ that is ſet forth in art. 13, may, perhaps, be thought to be in ſome degree obſcure and unſatisfactory, becauſe the dividend is a ſet of quantities that, all taken together, are equal to nothing; and it may be ſaid, with ſome appearance of truth, "that operations performed upon nothing, or a non-entity, can lead to no rational concluſion." I will therefore now ſeparate the quantities that compoſe that dividend, and place ſome of them on the ſecond, or right-hand, ſide of the equation, and then divide

divide both ſides of the equation ſo prepared by the ſame trinomial quantity $e^4 - 4le^2 + 4l^2$ as before; by which means we ſhall obtain two quotients which will be,—not equal to nothing, as the former quotient was, but,—of finite magnitudes and equal to each other: and then, I apprehend, no doubt can be entertained of the juſtneſs of the reaſoning and the truth of the concluſion obtained by it. This may be done in the following manner:

The compound quantity

$$\left\{\begin{array}{rrrrrr} e^{12} - 9le^{10} + 33l^2e^8 & - 63l^3e^6 & + 66l^4e^4 & - 36l^5e^2 \\ - mne^8 & + 9lmne^6 & + 4lm^3e^4 & + 36l^3mne^2 \\ - m^3e^6 & + 2m^2n^2e^4 & - 4l^2m^3e^2 \\ - n^3e^6 & - 28l^2mne^4 & - 4l^2n^3e^2 \\ & + 4ln^3e^4 & - 8lm^2n^2e^2 \\ & & + 8l^6 \\ & & + 8l^2m^2n^2 \\ & & - 16l^4mn \end{array}\right\}$$

is equal to the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ together with the multinomial quantity

$$\left\{\begin{array}{rrrrrr} - 5le^{10} + 29l^2e^8 & - 63l^3e^6 & + 66l^4e^4 & - 36l^5e^2 \\ - mne^8 & + 9lmne^6 & + 4lm^3e^4 & + 36l^3mne^2 \\ - m^3e^6 & + 2m^2n^2e^4 & - 4l^2m^3e^2 \\ - n^3e^6 & - 28l^2mne^4 & - 4l^2n^3e^2 \\ & + 4ln^3e^4 & - 8lm^2n^2e^2 \\ & & + 8l^6 \\ & & + 8l^2m^2n^2 \\ & & - 16l^4mn. \end{array}\right\}$$

Therefore the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ will be equal to the said multinomial quantity when the signs of all it's terms shall be changed, that is, to the following multinomial quantity, to wit,

$$\left\{\begin{array}{llllll}
+5le^{10} & -29l^2e^8 & +63l^3e^6 & -66l^4e^4 & +36l^5e^2 & \\
 & +mne^8 & -9lmne^6 & -4lm^3e^4 & -36l^3mne^2 & \\
 & & +m^3e^6 & -2m^2n^2e^4 & +4l^2m^3e^2 & \\
 & & +n^3e^6 & +28l^2mne^4 & +4l^2n^3e^2 & \\
 & & & -4ln^3e^4 & +8lm^2n^2e^2 & \\
 & & & & & -8l^6 \\
 & & & & & -8l^2m^2n^2 \\
 & & & & & +16l^4mn.
\end{array}\right.$$

For otherwise the former multinomial quantity, which consisted of the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ with this latter multinomial quantity subtracted from it, could not have been equal to nothing.

Now let both sides of this equation be divided by the trinomial quantity $e^4 - 4le^2 + 4l^2$: and it will follow that the quotients of these two divisions must be equal to each other.

But the quotient of the division of the trinomial quantity $e^{12} - 4le^{10} + 4l^2e^8$ by the trinomial quantity $e^4 - 4le^2 + 4l^2$ is e^8. And the quotient of the division of the second, or right-hand, side of this equation, or of the last-mentioned multinomial quantity, may be found in the following manner;

The

The Diviſion of the Compound Quantity, which forms the ſecond, or right-hand, Side of the laſt Equation, by the Trinomial Quantity $e^4 - 4le^2 + 4l^2$.

The Diviſor. The Dividend. The Quotient.

$e^4 - 4le^2 + 4l^2)$ $5le^{10} - 29l^2e^8 + 63l^3e^6 - 66l^4e^4 + 36l^5e^2$
$+ mne^8 - 9lmne^6 - 4lm^3e^4 - 36l^3mne^2 - 8l^6$
$+ m^3e^6 - 2m^2e^4 + 4l^2m^3e^2 - 8l^2m^2n^2$
$+ n^3e^6 + 28l^2mne^4 + 4l^2n^3e^2 + 16l^4mn$
$- 4ln^3e^4 + 8lm^2n^2e^2$

$+ 5le^6 - 9l^2e^4$
$+ mne^4$
$+ 7l^3e^2 - 2l^4$
$- 5lmne^2 + 4l^2mn$
$+ m^3e^2 - 2m^2n^2$
$+ n^3e^2$

The Firſt Subtrahend. $5le^{10} - 20l^2e^8 + 20l^3e^6$

$- 9l^2e^8 + 43l^3e^6 - 66l^4e^4$
$+ mne^8 - 9lmne^6 - 4lm^3e^4$
$+ m^3e^6 - 2m^2n^2e^4$
$+ n^3e^6 + 28l^2mne^4$
$- 4ln^3e^4$

The Second Subtrahend. $- 9l^2e^8 + 36l^3e^6 - 36l^4e^4$
$+ mne^8 - 4lmne^6 + 4l^2mne^4$

* $+ 7l^3e^6 - 30l^4e^4 + 36l^5e^2$
$- 5lmne^6 + 24l^2mne^4 - 36l^3mne^2$
$+ m^3e^6 - 4lm^3e^4 + 4l^2m^3e^2$
$+ n^3e^6 - 4ln^3e^4 + 4l^2n^3e^2$
$- 2m^2n^2e^4 + 8lm^2n^2e^2$

The Third Subtrahend.

$+ 7l^3e^6 - 28l^4e^4 + 28l^5e^2$
$- 5lmne^6 + 20l^2mne^4 - 20l^3mne^2$
$+ m^3e^6 - 4lm^3e^4 + 4l^2m^3e^2$
$+ n^3e^6 - 4ln^3e^4 + 4l^2n^3e^2$

* $- 2l^4e^4 + 8l^5e^2 - 8l^6$
$+ 4l^3mne^4 - 16l^3mne^2 - 8l^2m^2n^2$
$- 2m^2n^2e^4 + 8lm^2n^2e^2 + 16l^4mn$

The Fourth Subtrahend.

$- 2l^4e^4 + 8l^5e^2 - 8l^6$
$+ 4l^3mne^4 - 16l^3mne^2 + 16l^4mn$
$- 2m^2n^2e^4 + 8lm^2n^2e^2 - 8l^2m^2n^2$

* * *

It appears therefore that the quotient of the diviſion of this laſt multinomial quantity by the trinomial quantity $e^4 - 4le^2 + 4l^2$ is the compound quantity

$$\left\{\begin{array}{llll} 5le^6 - 9l^2e^4 & + 7l^3e^2 & - 2l^4 & \\ + mne^4 & - 5lmne^2 & + 4l^2mn & \\ & + m^2e^2 & - 2m^2n^2. & \\ & + n^2e^2 & & \end{array}\right\},$$

Therefore this compound quantity will be equal to e^8, which is the quotient of the diviſion of the quantity $e^{12} - 4le^{10} + 4l^2e^8$ by the ſame diviſor $e^4 - 4le^2 + 4l^2$; and therefore we have now obtained by intelligible means an equation between two finite quantities, to wit, the equation $e^8 =$

$$\left\{\begin{array}{llll} 5le^6 - 9l^2e^4 & + 7l^3e^2 & - 2l^4 & \\ + mne^4 & - 5lmne^2 & + 4l^2mn & \\ & + m^2e^2 & - 2m^2n^2, & \\ & + n^2e^2 & & \end{array}\right.$$

which, by ſubtracting the ſecond, or right-hand, ſide of it from e^8, or the left-hand ſide of it, will produce the former equation obtained by Dr. Wallis, to wit, the equation

$$\left.\begin{array}{llll} e^8 - 5le^6 + 9l^2e^4 & - 7l^3e^2 & + 2l^4 & \\ - mne^4 & + 5lmne^2 & - 4l^2mn & \\ & - m^2e^2 & + 2m^2n^2 & \\ & - n^2e^2 & & \end{array}\right\} = 0.$$

The Application of the Number 12.756,441,794,480,744, *or the ſecond Value of* ee, *or the leaſt Value of it but one, in the Equation* $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, *to the Solution of Colonel Titus's Problem.*

Art. 16. Having now at laſt, after many tedious and laborious Algebräick operations, obtained the equation

$$\left\{\begin{array}{llll} e^8 - 5le^6 + 9l^2e^4 & - 7le^2 & + 2l^4 \\ - mne^4 & + 5lmne^2 & - 4l^2mn \\ & - m^2e^2 & + 2m^2n^2 \\ & - n^2e^2 \end{array}\right\} = 0,$$

or (by ſubſtituting the values of l, m, and n in the terms of the equation inſtead of thoſe letters themſelves, and making the proper additions and ſubtractions of the terms that involve the ſame powers of ee) the equation $e^8 - 80e^6 + 1998e^4 - 14{,}937e^2 + 5000 = 0$, or the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, we will now proceed to apply one of the values of ee to the diſcovery of the value of aa, or $\frac{ee}{2}$, and of a, the firſt of the three unknown quantities a, b, and c, which are required to be found in Colonel Titus's Problem.

Now, if we ſuppoſe ee to be = 12.756,441,794,480,744, (which is the ſecond, or leaſt but one, of the four values of ee in this equation that have been found in the foregoing Appendix,) this value of ee will enable us to find the values of the ſaid three numbers a, b, and c required by the Problem.

For, if ee is = 12.756,441,794,480,744, we ſhall have $aa \left(= \frac{ee}{2} = \frac{12.756,441,794,480,744}{2}\right) = 6.378,220,897,$

897,240,372, and consequently a (= $\sqrt{6.378,220,897,240,372}$) = 2.525,513,986. Therefore a, or the first and least of the three unknown quantities a, b, and c required to be found in the said Problem, will be = 2.525,513,986. Q. E. I.

Having thus found the value of a, or the first of the three numbers a, b, and c, we may derive from it the value of b, or the second of those numbers, in the manner following:

It has been shewn above in art. 8 that b is = $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$.

Now aa is = 6.378,220, &c, and consequently a^4 will be (= $\overline{6.378,220, \&c}^2$) = 40.681,690,368,400, and $8na^4$ will be (= 8 × 18 × a^4 = 144 × a^4 = 144 × 40.681,690,368,400) = 5858.163,413,049,600; and m^2a^2 will be (= 17 × 17 × a^2 = 289 × a^2 = 289 × 6.378,220, &c) = 1843.305,580, &c; and $12lna^2$ will be (= 12 × 16 × 18 × a^2 = 3456 × a^2 = 3456 × 6.378,220, &c) = 22,043.128,320; and $4l^2n$ will be (= 4 × 16 × 16 × 18 = 4 × 256 × 18 = 1024 × 18) = 18,432. Therefore the quadrinomial quantity $8na^4 - 12lna^2 + m^2a^2 + 4l^2n$ will be (= 5858.163,413,049,600 — 22,043.128,320 + 1843.305,580, &c + 18,432 = 26,133.468,993,049,600 — 22,043.128,320) = 4090.340,673,049,600. Therefore $\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}$ will be (= $\sqrt{4090.340,673,049,600}$) = 63 955,771; and $\frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ will be (= $\frac{63.955,771}{2n}$ =

$= \frac{63.955,771}{2 \times 18} = \frac{63.955,771}{36}$) 1.776,549. And $\frac{ma}{2n}$ will be ($= \frac{17a}{36} = \frac{17 \times 2.525,513,986}{36} = \frac{42.933,737,762}{36}$) $= 1.192,603,826$. Therefore $\frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$ will be ($= 1.192,603,826 + 1.776,549$) $= 2.969,152,826$, or, nearly, 2.969,153; that is, b, or the ſecond of the unknown quantities a, b, and c, required to be found, will be $=$, nearly, 2.969,153. Q. E. I.

Laſtly, ſince $aa + bc$ is $= l$, we ſhall have $bc = l - aa$, and conſequently $c = \frac{l-aa}{b}$ ($= \frac{16-6.378,220,8}{2.969,153} = \frac{9.621,780}{2.969,153}$) $= 3.240,580$; that is, the third unknown number c which was required to be found will be $=$ 3.240,580. Therefore the three numbers ſought are 2.525,513,986, &c, or, nearly, 2.525,514, and 2.969,153, and 3.240,580. Q. E. I.

Art. 17. Theſe numbers will anſwer the conditions of the problem. For, if a is $=$ 2.525,514, and b is $=$ 2.969,153, and c is $=$ 3.240,580, we ſhall have $aa =$ 6.378,220,8, and $bb =$ 8.815,869,5, and $cc =$ 10.501, 358,7, and bc ($=$ 2.969,153 $\times$ 3.240,580) $=$ 9.621, 778,8, and ac ($=$ 2.525,514 $\times$ 3.240,580,) $=$ 8.184, 130,1, and ab ($=$ 2.525,514 $\times$ 2.969,153) $=$ 7.498, 637,4. Therefore $aa + bc$ will be ($= 6.378,220,8 +$ 9.621,778,8) $= 15.999,998,6$; which is very nearly equal to 16, agreeably to the firſt condition of the Problem.

blem. And $bb + ac$ will be $(= 8.815,869,5 + 8.184,130,1) = 16.999,999,6$; which is very nearly equal to 17, agreeably to the ſecond condition of the Problem. And $cc + ab$ will be $(= 10.501,358,7 + 7.498,637,4) = 17.999,996,1$; which is very nearly equal to 18, agreeably to the third condition of the Problem. Q. E. D.

Art. 18. Dr. Wallis has inveſtigated the values of the numbers a, b, and c to ſixteen places of figures, and found them and their ſquares and products to be as follows; to wit,

$a = 2.525,513,986,744,158$, and $aa = 6.378,220,897,240,372$,

$b = 2.969,152,768,619,848$, and $bb = 8.815,868,163,402,909$,

$c = 3.240,580,681,617,174$, and $cc = 10.501,363,154,070,430$,

and $bc = 9.621,779,102,759,628$, and $ac = 8.184,131,836,597,093$,

and $ab = 7.498,636,845,929,567$; and conſequently

$aa + bc$ to be $=$

$$\left\{ \begin{array}{r} 6.378,220,897,240,372, \\ + \ 9.621,779,102,759,628 \end{array} \right\}$$

$= 16.000,000,000,000,000$, agreeably to the firſt condition of the Problem;

and $bb + ac$ to be $=$

$$\left\{ \begin{array}{r} 8.815,868,163,402,909 \\ + \ 8.184,131,836,597,093 \end{array} \right\}$$

$= 17.000,000,000,000,002$; which is very

nearly equal to 17, agreeably to the fecond condition of the Problem;

and $cc + ab$ to be =

$$\left\{ \begin{array}{r} 10.501,363,154,070,430 \\ + \ 7.498,636,845,929,567 \end{array} \right\}$$

= 17.999,999,999,999,997, which is very nearly equal to 18, agreeably to the third condition of the Problem.

The Application of the Number 0.350,987,045,866,14, *or the leaſt Value of* ee *in the Equation* $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, *to the Solution of Colonel Titus's Problem.*

Art. 19. But there is another ſet of values of the unknown numbers a, b, and c that will anſwer the conditions of the Problem. And theſe values may be derived from the leaſt value of ee in the final equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$ reſulting from the foregoing ſolution of the Problem. For the leaſt value of ee in that equation, or of x in the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, has been found in the foregoing Appendix to Dr. Halley's Tract, page 160, to be = 0.350,986,9, or nearly, 0.350,987; and, if it had been inveſtigated to a greater degree of exactneſs, it would have been found to be = 0.350,987,045,866,14. Therefore, if we take ee = 0.350,987,045,866,14, we ſhall have $aa \left(= \frac{ee}{2} = \frac{0.350,987,045,866,14}{2}\right) =$ 0.175,

0.175,493,522,933.07, and consequently a (= $\sqrt{0.175,493,522,933,07}$) = 0.418,919,470,701,7; that is, the firſt and leaſt of the three numbers a, b, and c which are required to be found, will be = 0.418,919,470,701,7, or nearly, 0.418,919,470. Q. E. I.

Further, it has been ſhewn above in art. 8 that the ſecond unknown number b is $= \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12mna^2 + m^2a^2 + 4l^2n}}{2n} = \frac{17a}{2 \times 18}$

$$+ \frac{\sqrt{8 \times 18a^4 - 12 \times 16 \times 18a^2 + 17 \times 17a^2 + 4 \times 16 \times 16 \times 18}}{2 \times 18}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 12 \times 289a^2 + 289a^2 + 1024 \times 18}}{36}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3456a^2 + 289a^2 + 18,432}}{36}$$

$$= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}.$$

But a is = 0.418,919,470, and aa is = 0.175,493,523, and consequently a^4 is $(= \overline{0.175,493,523}^2)$ = 0.030,797,976,63.

Therefore $144a^4$ will be (= 144 × 0.030,797,976,63) = 4.434,908,634,72, and $3167a^2$ will be (= 3167 × 0.175,493,523) = 555.787,987,341, and $144a^4 - 3167a^2 + 18,432$ will be (= 4.434,908,634,72 — 555.787,987,341 + 18,432 = 18,436.434,908,634,7 — 555.787,

555.787,987,341) = 17,880.646,921,293,72, and the ſquare-root of $144a^4 - 3167a^2 + 18,432$ will be (= $\sqrt{17,880.646,921,293,72}$) = 133.718,536,191;

and $\frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$ will be

$(= \frac{133.718,536,191}{36}) = 3.714,403,783.$

And $\frac{17a}{36}$ will be $(= \frac{17 \times 0.418,919,470}{36} = \frac{7.121,630,990}{36}) = 0.197,823,083.$

Therefore $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18432}}{36}$ will be (= 0.197,823,083 + 3.714,403,783) = 3.912,226,866; that is, b, or the ſecond unknown quantity required to be found, will be = 3.912,226,866. Q. E. I.

Laſtly, the third number c will be = $\frac{1 - aa}{b}$ (= $\frac{16 - 0.175,493,523}{3.912,226,866} = \frac{15.824,506,477}{3.912,226,866}$) = 4.044,884,670. Q. E. I.

Art. 20. And theſe three numbers, 0.418,919,470, 3.912,226,866, and 4.044,884,670, will anſwer the conditions of the Problem. For, if a is = 0.418,919,470, and b is = 3.912,226,866, and c is = 4.044,884,670, we ſhall have

aa (=

aa $(= \overline{0.418,919,470}|^2) = 0.175,493,523,$

and bb $(= \overline{3.912,226,866}|^2) = 15.305,519,051,$

and cc $(= \overline{4.044,884,670}|^2) = 16.361,091,993,$

and bc $(= 3.912,226,866 \times 4.044,884,670) = 15.824,506,475,$

and ac $(= 0.418,919,470 \times 4.044,884,670) = 1.694,480,942,$

and ab $(= 0.418,919,470 \times 3.912,226,866) = 1.638,908,005,$

Therefore $aa + bc$ will be =

$$\left\{ \begin{array}{r} 0.175,493,523 \\ + \ 15.824,506,475 \end{array} \right\}$$

$= 15.999,999,998$; which is very nearly $= 16$, agreeably to the firſt condition of the Problem:

And $bb + ac$ will be =

$$\left\{ \begin{array}{r} 15.305,519,051 \\ + \ 1.694,480,942 \end{array} \right\}$$

$= 16.999,999,993$; which is very nearly equal to 17, agreeably to the ſecond condition of the Problem:

And $cc + ab$ will be =

16.361,

$$\left\{ \begin{array}{r} 16.361,091,993 \\ + \ 1.638,9c8,005 \end{array} \right\}$$

$= 17.999,999,998$; which is very nearly equal to 18, agreeably to the third condition of the Problem.

It appears therefore that theſe three numbers 0.418, 919,470, 3.912,226,866, and 4.044,884,670, which are derived from 0.350,987,046, or the firſt, or leaſt, value of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, anſwer the conditions of the Problem as well as the three former numbers 2.525,513,986, and 2.969,153, and 3.240,580, which are derived from 12.756,441,794, 480,744, or the ſecond value of ee, or the leaſt value of ee but one, in the ſame equation.

Of the Two Greateſt Values of ee *in the Equation* $14{,}937ee - 1998e^4 + 8ce^6 - e^4 = 5000$.

Art. 21. But the other two values of ee in the equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5c00$, to wit, the numbers 32.060,290,8 and 34.832,280,2, have no relation to the preſent Problem: for, if the values of a, b, and c are derived from them by taking $aa = \frac{ee}{2}$, upon a ſuppoſition that ee is equal to either of thoſe two numbers, and taking $b = \frac{ma}{2n} + \frac{\sqrt{8na^4 - 12lna^2 + m^2a^2 + 4l^2n}}{2n}$, or $\frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and $c = \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the values of $aa + bc$, $bb + ac$, and

cc

$cc + ab$ will not be found to be reſpectively equal to the three numbers 16, 17, and 18, agreeably to the conditions of the Problem. But theſe two greater values of ee in the ſaid equation $14{,}937ee - 1998e^4 + 80e^6 - e^8 = 5000$, or theſe two greateſt roots of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, will be found to relate to two other Problems, ſomewhat different from that of Colonel Titus. For, if ee is taken = 32.060,290,8, and aa is taken $= \frac{ee}{2}$, or $\frac{32.060,290,8}{2}$, or 16.030,145,4, and $a = \sqrt{16.030,145,4}$, and b is taken $= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and $c = \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the three values of a, b, and c ſo obtained will be fitted to anſwer the conditions of a Problem in which it ſhould be required to find the values of three unknown numbers a, b, and c, upon a ſuppoſition that $aa - bc$ (inſtead of $aa + bc$) was equal to 16, and that $bb + ac$ was (as before) equal to 17, and that $cc - ab$ (inſtead of $cc + ab$) was equal to 18. And, if ee is taken = 34.832,280,2, and aa is taken $= \frac{ee}{2}$, or $\frac{34.832,280,2}{2}$, or 17.416,140,1, and b is taken $= \frac{17a}{36} + \frac{\sqrt{144a^4 - 3167a^2 + 18,432}}{36}$, and c is taken $= \frac{l - aa}{b}$, or $\frac{16 - aa}{b}$, the three values of a, b, and c ſo obtained will be fitted to anſwer the conditions of a third

third Problem, in which it ſhould be required to find the values of three unknown numbers a, b, and c, upon a ſuppoſition that $aa - bc$ (inſtead of $aa + bc$) was equal to 16, and that $bb - ac$ (inſtead of $bb + ac$) was equal to 17, and that $cc + ab$ was (as before) equal to 18. See Dr. Wallis's Algebra, chapter LXII, articles 63, 64, 65, 66, &c - - 79, in which the ſubject is fully diſcuſſed, but not without ſome degree of obſcurity ariſing from the conſideration of negative quantities, and from the doctrine of the generation of equations one from another by multiplication, or by bringing all the terms of each equation to the firſt, or left-hand, ſide of the equation, ſo as to make them equal to 0, and then multiplying the equations (ſo prepared and made equal to nothing,) one into another, and from the eminently falſe poſition derived from that manner of generating equations, to wit, " that every Algebräick equation has as many roots as there are units in the index of the higheſt power of the unknown quantity contained in the equation." For this doctrine of the generation of equations one from another by multiplication, (which was invented by Harriot, and adopted by Des Cartes and Dr. Wallis and almoſt all the ſubſequent writers on Algebra,) inſtead of being an improvement in that ſcience, has, in my opinion, been of great detriment to it by deſtroying it's ſimplicity and perſpicuity, and therefore ought again to be diſcarded from it.

Art. 22. The foregoing Solution of Colonel Titus's Problem given us by Dr. Wallis is, as we have ſeen, exceedingly tedious and laborious; and great part of the

labour required in the ſolution of it ariſes from the neceſſity we are under of raiſing the equation involving the unknown quantity a to the 12th order, or to the 12th power of a, in order to free it's terms from radicality. But Mr. William Frend, the ingenious author of the late perſpicuous Treatiſe on Algebra in one volume octavo, intitled *Principles of Algebra*, (in which he totally rejects the abſurd and perplexing doctrine of *negative quantities*, or *quantities leſs than nothing*, or *quantities obtained by ſubtracting a greater quantity from a leſſer*,) has lately communicated to me another Solution of this Problem, which produces only a biquadratick equation. And this equation will be found to have three roots, or, in the language of modern Algebraiſts, three real and affirmative roots; of which the middle root will enable us to find the firſt ſet of values of the three unknown numbers a, b, and c, that will anſwer the conditions of the Problem, to wit, the three numbers 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c; and the greateſt root will enable us to find the ſecond ſet of values of the ſaid three unknown numbers that will anſwer the ſame conditions, to wit, the three numbers 0.418,919,47, 3.912,226,8 &c, and 4.044,884,6 &c. This Solution I ſhall now proceed to lay before the reader, as a proper accompaniment to the foregoing Solution of this Problem given us by Dr. Wallis.

ANOTHER

ANOTHER SOLUTION

OF

COLONEL TITUS's PROBLEM,

BY MR. WILLIAM FREND, M. A.

FELLOW OF JESUS COLLEGE, CAMBRIDGE.

Art. 23. RETAINING the notation uſed in the foregoing Solution of this Problem by Dr. Wallis, and retaining likewiſe the preliminary obſervations made by Dr. Wallis, to wit, "that the three unknown numbers a, b, and c, that are ſought in the Problem, are, all of them, of different magnitudes, and that a, or the number of which the ſquare occurs in the firſt equation $aa + bc = 16$, is the leaſt of the three, and that c, or the number of which the ſquare occurs in the third equation $cc + ab = 18$, is the greateſt of the three," let it be ſuppoſed that the ſecond unknown number b is greater than the firſt unknown number a in the proportion of x to 1, and that the third unknown number c is greater than the firſt unknown number in the proportion of y to 1.

Then will b be $= xa$, and c will be $= ya$, and conſequently bb will be $= x^2a^2$, and cc will be $= y^2a^2$, and

bc

bc will be $(= xa \times ya) = xyaa$, and ac will be $(= a \times ya) = yaa$, and ab will be $(= a \times xa) = xaa$. Therefore the firſt equation $aa + bc = 16$, or $aa + bc = l$, will now become $aa + xyaa = 16$, or $aa + xyaa = l$; and the ſecond equation $bb + ac = 17$, or $bb + ac = m$, will now become $x^2a^2 + ya^2 = 17$, or $x^2a^2 + ya^2 = m$; and the third equation $cc + ab = 18$, or $cc + ab = n$, will now become $y^2a^2 + xa^2 = 18$, or $y^2a^2 + xa^2 = n$.

Art. 24. Now, ſince $aa + xyaa$ is $= l$, we ſhall have $aa = \frac{l}{1+xy}$. And, ſince $x^2a^2 + ya^2$ is $= m$, we ſhall have $aa = \frac{m}{x^2+y}$. And, ſince $y^2a^2 + xa^2$ is $= n$, we ſhall have $aa = \frac{n}{yy+x}$.

Further, ſince aa is $= \frac{l}{1+xy}$, and aa is alſo $= \frac{m}{x^2+y}$, we ſhall have $\frac{l}{1+xy} = \frac{m}{x^2+y}$; and conſequently, (multiplying both ſides by $\overline{1+xy} \times \overline{x^2+y}$) we ſhall have $lx^2 + ly = m + mxy$; and (ſubtracting m from both ſides,) $lx^2 + ly - m = mxy$, and (ſubtracting ly from both ſides) $lx^2 - m = mxy - ly$, and (dividing both ſides by $mx - l$) we ſhall have $y = \frac{lx^2 - m}{mx - l}$. And thus we have obtained a value of y expreſſed by a fraction which involves only the known

quantities l and m and the unknown quantity x. We muſt now endeavour to find another value of y expreſſed, like the former, in terms that involve only known quantities and the ſame unknown quantity x. And this may be done as follows:

Art. 25. Since aa is $= \frac{m}{x^2 + y}$, and aa is likewiſe $= \frac{n}{yy + x}$, it follows that $\frac{m}{x^2 + y}$ will be $= \frac{n}{yy + x}$. Therefore (multiplying both ſides into $\sqrt{x^2 + y} \times \sqrt{y^2 + x}$) we ſhall have $myy + mx = nx^2 + ny$; and (ſubtracting mx from both ſides,) we ſhall have $myy = nxx - mx + ny$, and (ſubtracting ny from both ſides,) $myy - ny = nxx - mx$, and (dividing both ſides by m) $yy - \frac{ny}{m} = \frac{nxx - mx}{m}$. Therefore (adding $\frac{n^2}{4m^2}$ to both ſides,) we ſhall have $yy - \frac{ny}{m} + \frac{n^2}{4m^2} = \frac{nxx - mx}{m} + \frac{n^2}{4m^2}$; and (extracting the ſquare-roots of both ſides) we ſhall have $y - \frac{n}{2m} = \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}}$, and (adding $\frac{n}{2m}$ to both ſides,) $y = \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}} + \frac{n}{2m}$.

Art. 26. But it was before ſhewn that y is $= \frac{lx^2 - m}{mx - l}$.

Therefore

Therefore $\frac{lx^2 - m}{mx - l}$ will be $= \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}} + \frac{n}{2m}$; and conſequently $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ will be $= \sqrt{\frac{nxx - mx}{m} + \frac{n^2}{4m^2}}$, and therefore the ſquare of $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ will be equal to $\frac{nxx - mx}{m} + \frac{n^2}{4m^2}$.

But the ſquare of $\frac{lx^2 - m}{mx - l} - \frac{n}{2m}$ is $= \overline{\frac{lx^2 - m}{mx - l}}^2 + \overline{\frac{n}{2m}}^2 - 2 \times \frac{n}{2m} \times \frac{lx^2 - m}{mx - l} = \frac{\overline{lx^2 - m}^2}{\overline{mx - l}^2} + \frac{n^2}{4m^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l} = \frac{l^2x^4 - 2lmx^2 + m^2}{\overline{mx - l}^2} + \frac{n^2}{4m^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l}$.

Therefore $\frac{l^2x^4 - 2lmx^2 + m^2}{\overline{mx - l}^2} + \frac{n^2}{4m^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l}$ will be $= \frac{nxx - mx}{m} + \frac{n^2}{4m^2}$. And conſequently (ſubtracting $\frac{n^2}{4m^2}$ from both ſides,) we ſhall have $\frac{l^2x^4 - 2lmx^2 + m^2}{\overline{mx - l}^2} - \frac{n}{m} \times \frac{lx^2 - m}{mx - l} = \frac{nxx - mx}{m}$, and (multiplying both ſides into m)

$\frac{l^2mx^4 - 2lm^2x^2 + m^3}{mx - l^2} - n \times \frac{lx^2 - m}{mx - l} = nxx - mx$;

and (multiplying both ſides into $mx - l^2$, or $m^2x^2 - 2lmx + l^4$,) we ſhall have $l^3mx^4 - 2lm^2x^2 + m^3 - n \times lx^2 - m \times mx - l = nxx - mx \times m^2x^2 - 2lmx + l^2$, or $l^3mx^4 - 2lm^2x^2 + m^3 - n \times lmx^3 - l^2x^2 - m^2x + lm = m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$, or $l^3mx^4 - 2lm^2x^2 + m^3 - lmnx^3 + l^2nx^2 + m^2nx - lmn = m^2nx^4 - 2lmnx^3 + l^2nx^2 - m^3x^3 + 2lm^2x^2 - l^2mx$, and (ſubtracting l^2nx^2 from both ſides,) $l^3mx^4 - 2lm^2x^2 + m^3 - lmnx^3 + m^2nx - lmn = m^2nx^4 - 2lmnx^3 - m^3x^3 + 2lm^2x^2 - l^2mx$, and (adding $lmnx^3 + 2lm^2x^2$ to both ſides) $l^3mx^4 + m^3 + m^2nx - lmn = m^2nx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx$, and (ſubtracting $l^3mx^4 + m^2nx$ from both ſides,) $m^3 - lmn = m^2nx^4 - l^3mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx$, or $m^2nx^4 - l^3mx^4 - lmnx^3 - m^3x^3 + 4lm^2x^2 - l^2mx - m^2nx = m^3 - lmn$, and (dividing all the terms by m) $mnx^4 - l^2x^4 - lnx^3 - m^2x^3 + 4lmx^2 - l^2x - mnx = m^2 - ln$, or $mn - l^2 \times x^4 - ln + m^2 \times x^3 + 4lmx^2 - l^2 + mn \times x = m^2 - ln$; and, laſtly, (dividing all the terms by $mn - l^2$) we ſhall have $x^4 - \frac{ln + m^2}{mn - l^2} \times x^3 + \frac{4lm}{mn - l^2} \times x^2 - \frac{l^2 + mn}{mn - l^2} \times x = \frac{m^2 - ln}{mn - l^2}$.

Art. 27. But, becauſe l is $= 16$, and m is $= 17$, and n is $= 18$, we ſhall have $ln\ (= 16 \times 18) = 288$, and

mn

mn ($= 17 \times 18$) $= 306$, and l^2 ($= 16 \times 16$) $= 256$, and m^2 ($= 17 \times 17$) $= 289$. Therefore $\frac{ln + m^2}{mn - l^2}$ will be ($= \frac{288 + 289}{306 - 256} = \frac{577}{50} = \frac{1154}{100}$) $= 11.54$; and $\frac{4lm}{mn - l^2}$ will be ($= \frac{4 \times 16 \times 17}{306 - 256} = \frac{4 \times 272}{50} = \frac{1088}{50} = \frac{2176}{100}$) $= 21.76$; and $\frac{l^2 + mn}{mn - l^2}$ will be ($= \frac{256 + 306}{306 - 256} = \frac{562}{50} = \frac{1124}{100}$) $= 11.24$; and $\frac{m^2 - ln}{mn - l^2}$ will be $= \frac{289 - 288}{306 - 256} = \frac{1}{50} = \frac{2}{100}$) $= 0.02$.

Therefore the equation $x^4 - \left|\frac{ln + m^2}{mn - l^2}\right| \times x^3 + \frac{4lm}{mn - l^2} \times x^2 - \left|\frac{l^2 + mn}{mn - l^2}\right| \times x = \frac{m^2 - ln}{mn - l^2}$ will, when theſe ſubſtitutions of the ~~original~~ numbers 16, 17, and 18 for the letters l, m, and n have been made in it, become $x^4 - 11.54 \times x^3 + 21.76 \times x^2 - 11.24 \times x = 0.02$. This equation muſt therefore be reſolved in order to obtain the value of x; and from the value of x ſo obtained, or, if the equation ſhould have more than one root, (as will be found to be the caſe,) from that value of x which has a relation to the preſent Problem, we muſt then derive the value of y, expreſſed by it's relation to x, by computing the fraction $\frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$, to which we have ſeen, in art. 24, that y is equal. And, when this is done, we muſt multiply the value of xa, or b, into the

value of ya, or c, which will give us the value of bc expreſſed by it's relation to aa; and then the equation $aa + bc = 16$ will be a quadratick equation involving only one unknown quantity, namely, a, and which may therefore be eaſily reſolved; and the reſolution of it will give us the value of a, or the firſt of the three unknown quantities a, b, and c, that the Problem requires us to find. And from the value of a ſo found we may derive the values of b and c by multiplying a into x to obtain the number b, and by multiplying a into y to obtain the number c. Our next buſineſs therefore muſt be to find the roots of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

The Reſolution of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 28. Let us, firſt, ſuppoſe that x is $= 1$.

And we ſhall then have $x^2 = 1$, and $x^3 = 1$, and $x^4 = 1$; and conſequently $x^4 + 21.76x^2$ will be $(= 1 + 21.76) = 22.76$, and $11.54x^3 + 11.24x$ will be $(= 11.54 + 11.24) = 22.78$; which is greater than 22.76, and therefore cannot be ſubtracted from it. Therefore in this caſe the binomial quantity $11.54x^3 + 11.24x$ is greater than the binomial quantity $x^4 + 21.76x^2$, and cannot be ſubtracted from it, and conſequently the quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ (which

(which ſuppoſes the ſubtraction of $11.54x^3 + 11.24x$ from $x^4 + 21.76x^2$) cannot exiſt. And the ſame thing will be true, if x be of any magnitude leſs than 1. But, when x is a little greater than 1, the binomial quantity $x^4 + 21.76x^2$ (which before had been leſs than the binomial quantity $11.54x^3 + 11.24$) will become equal to the binomial quantity $11.54x^3 + 11.24x$, and the whole quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $= 0$; and, when x increaſes ſtill further, the ſaid quadrinomial quantity will increaſe from 0 to a certain quantity, after which it will decreaſe to a certain quantity, and then, finally, it will increaſe from that quantity *ad infinitum*, which makes it become at three different inſtants of time equal to the ſame quantity. Thus, for example, when x is become $= 1 + \frac{1}{10}$, or 1.1, we ſhall have $x^2 = 1.21$, and $x^3 = 1.331$, and $x^4 = 1.14641$, and $11.24x$ $(= 11.24 \times 1.1) = 12.364$, and $21.76x^2$ $(= 21.76 \times 1.21) = 26.3296$, and $11.54x^3$ $(= 11.54 \times 1.331) = 15.359,74$; and conſequently the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.4641 - 15.359,74 + 26.3296 - 12.364 = 27.793,70 - 27.723,74) = 0.069,96$. Therefore, while x increaſes from the magnitude which it has when $x^4 + 21.76x^2$ is $= 11.54x^3 + 11.24x$, or when the quadrinomial quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0$, to 1.1, the ſaid quadrinomial quantity will increaſe from 0 to 0.069,96, and conſequently it will at ſome former inſtant of time during it's ſaid increaſe have been equal to 0.2, which is leſs than 0.069,96; or, in other words, the leaſt

root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be leſs than 1.1, but greater than 1.

Art. 29. Since we now know that the ſaid root is leſs than 1.1, but greater than 1, we will, in the next place, ſuppoſe it to be nearly = 1.03, and will try the effect of this ſuppoſition.

Now, if x is = 1.03, we ſhall have xx (= $\overline{1.03}^2$) = 1.0609, and x^3 (= $\overline{1.03}^3$) = 1.092,727, and x^4 (= $\overline{1.03}^4$) = 1.125,508,81. Therefore $11.24x$ will be (= 11.24 × 1.03) = 11.5772; and $21.76x^2$ will be (= 21.76 × 1.0609) = 23.085,184; and $11.54x^3$ will be (= 11.54 × 1.092,727) = 12.610,069,58; and conſequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be (= 1.125,508,81 − 12.610,069,58 + 23.085,184,00 − 11.577,200,00 = 24.210,692,81 − 24.187,269,58) = 0.023,423,23; which (though much leſs than 0.069,96, or the reſult of the ſubſtitution of 1.1 inſtead of x in the ſaid compound quantity) is ſtill a little greater than 0.02, or the abſolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. And conſequently 1.03 will be ſomewhat greater than the root of that equation.

Art. 30. We will therefore, in the third place, ſuppoſe x to be = 1.027 (inſtead of 1.03, or 1.030) and try the effect of that conjecture.

Now,

Now, if x is $= 1.027$, we ſhall have $x^2 (= \overline{1.027}^2) = 1.054,729$, and $x^3 (= \overline{1.027}^3) = 1.083,206,683$, and $x^4 (= \overline{1.027}^4) = 1.112,453,441$. Therefore $11.24x$ will be $(= 11.24 \times 1.027) = 11.543,48$, and $21.76x^2$ will be $(= 21.76 \times 1.054,729) = 22.950,903,04$, and $11.54x^3$ will be $(= 11.54 \times 1.083,206,683) = 12.500,205,121,82$; and conſequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.112,453,263,441 - 12.500,205,121,82 + 22.950,903,04 - 11.543,48 = 24.063,356,303,441 - 24.043,685,121,82) = 0.019,671,181,621$; which is a little leſs than 0.02, or the abſolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. Therefore 1.027 will be a little leſs than the true value of x in that equation. We may therefore now conclude that the ſaid true value is leſs than 1.03, or 1.030, but greater than 1.027, and likewiſe that it is nearer to 1.027 than to 1.03, or 1.030.

Art. 31. We will now make uſe of this laſt near value of x, to wit, 1.027, as a baſis for a further approach to it's true value by Mr. Raphſon's method of approximation, and for that purpoſe we will ſuppoſe x to be equal to $1.027 + z$, and will ſubſtitute that binomial quantity inſtead of x in the terms of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. This may be done as follows:

If x is $= 1.027 + z$, we ſhall have $x^2 (= \overline{1.027 + z}^2 = \overline{1.027}^2 + 2 \times 1.027 \times z + \&c = \overline{1.027}^2 + 2.054 \times z + \&c) = 1.054,729 + 2.054 \times z + \&c$,

and

and x^3 $(= \overline{1.027 + z}|^3 = \overline{1.027}|^3 + 3 \times \overline{1.027}|^2 \times z + \&c$
$= \overline{1.027}|^2 + 3 \times 1.054,729 \times z + \&c = \overline{1.027}|^3$
$+ 3.164,187 \times z + \&c) = 1.083,206,683 +$
$3.164,187 \times z + \&c,$

and x^4 $(= \overline{1.027 + z}|^4 = \overline{1.027}|^4 + 4 \times \overline{1.027}|^3 \times z + \&c$
$= \overline{1.027}|^4 + 4 \times 1.083,206,683 \times z + \&c =$
$\overline{1.027}|^4 + 4.332,826,732 \times z + \&c) = 1.112,453,$
$263,441 + 4.332,826,732 \times z + \&c.$

Therefore $11.24x$ will be $(= 11.24 \times \overline{1.027 + z} =$
$11.24 \times 1.027 + 11.24 \times z) = 11.543,48 + 11.24 \times z;$

And $21.76x^2$ will be $(= 21.76 \times$
$\overline{1.054,729 + 2.054 \times z + \&c}$
$= 21.76 \times 1.054,729 + 21.76 \times 2.054 \times z + \&c$
$= 21.76 \times 1.054,729 + 44.695,04 \times z + \&c)$
$= 22.950,903,04 \quad + 44.695,04 \times z + \&c;$

And $11.54x^3$ will be $(= 11.54 \times$
$\overline{1.083,206,683 + 3.164,187 \times z + \&c}$
$= 11.54 \times 1.083,206,683 + 11.54 \times 3.164,187 \times z + \&c$
$= 11.54 \times 1.083,206,683 + 35.565,461,88 \times z + \&c)$
$= 12.500,205,121,82 \quad + 35.565,461,88 \times z + \&c;$

and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $=$ to the compound quantity

1.112,

$$\left\{\begin{array}{llll} 1.112,453,263,441 & + \ 4.332,826,732 & \times z + \&c \\ - \ 12.500,205,121,82 & - \ 35.565,461,88 & \times z - \&c \\ + \ 22.950,903,04 & + \ 44.695,04 & \times z + \&c \\ - \ 11.543,48 & - \ 11.24 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{llll} 24.063,356,303,441 & + \ 49.027,866,732 & \times z + \&c \\ - \ 24.043,685,121,82 & - \ 46.805,461,88 & \times z - \&c \end{array}\right\}$$

$$= 0.019,671,181,621 + 2.222,404,852 \times z + \&c.$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0.02$.

Therefore the compound quantity $0.019,671,181,621 + 2.222,404,852 \times z$ will alſo be $= 0.02$. And conſequently, (ſubtracting $0.019,671,181,621$ from both ſides,) we ſhall have $2.222,404,852 \times z$ $(= 0.020,000,000 - 0.019,671,181) = 0.000,328,379$, and z $\left(= \frac{0.000,328,379}{2.222,404,852}\right) = 0.000,147,9$, or, nearly, $0.000,148$. Therefore x, or $1.027 + z$, will be $(= 1.027 + 0.000,148) = 1.027,148$; that is, the root of the equation $x^4 - 11.54x^3 + 21.76x^2 + 11.24x = 0.02$ will be, nearly, $= 1.027,148$. Q. E. I.

Art. 32. We will therefore, in the next place, ſuppoſe x to be $= 1.027,148$, and try the effect of this ſuppoſition.

Now,

Now, if x is $= 1.027,148$, we ſhall have x^2 $(= \overline{1.027,148}^2) = 1.055,033,013,904$, and x^3 $(= \overline{1.027,148}^3 = 1.055,033,0 \times 1.027,148) = 1.083,675,0$, and x^4 $(= 1.083,675,0 \times 1.027,148) = 1.113,194,6$. Therefore $11.24x$ will be $(= 11.24 \times 1.027,148) = 11.545,143,5$, and $21.76x^2$ will be $(= 21.76 \times 1.055,033,0) = 22.957,518,0$, and $11.54x^3$ will be $(= 11.54 \times 1.083,675,0) = 12.505,609,5$; and conſequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $(= 1.113,194,6 - 12.505,609,5 + 22.957,518,0 - 11.545,143,5 = 24.070,712,6 - 24.050,753,0) = 0.019,959,6$; which is a little leſs than 0.02, or the abſolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. Therefore 1.027,148 will be a little leſs than the true value of x in that equation.

Art. 33. We will now have recourſe to a ſecond proceſs of Mr. Raphſon's method of approximation, in order to obtain a more exact value of this root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, and for that purpoſe will ſuppoſe x to be $= 1.027,148 + z$, and will ſubſtitute $1.027,148 + z$ inſtead of it in the terms of the ſaid equation, and then reſolve the transformed equation reſulting from ſuch ſubſtitution, as if it were a mere ſimple equation, by omitting all the terms of it that involve any higher power of the unknown quantity z than the ſimple power, or z itſelf. This may be done in the manner following:

Since

Since x is $= 1.027,148 + z$, we shall have x^2 $(= \overline{1.027,148 + z}^2 = \overline{1.027,148}^2 + 2 \times 1.027,148 \times z + \&c = \overline{1.027,148}^2 + 2.054,296 \times z + \&c) = 1.055,033,013,904 + 2.054,296 \times z + \&c,$

and x^3 $(= \overline{1.027,148 + z}^3 = \overline{1.027,148}^3 + 3 \times \overline{1.027,148}^2 \times z + \&c$

$= \overline{1.027,148}^3 \times 3 \times 1.055,0330 \times z + \&c$

$= \overline{1.027,148}^3 + 3.165,099,0 \times z + \&c)$

$= 1.083,675,0 + 3.165,099,0 \times z + \&c,$

and x^4 $(= \overline{1.027,148 + z}^4 = \overline{1.027,148}^4 + 4 \times \overline{1.027,148}^3 \times z + \&c$

$= \overline{1.027,148}^4 + 4 \times 1.083,675,0 \times z + \&c$

$= \overline{1.027,148}^4 + 4.334,700,0 \times z + \&c)$

$= 1.113,194,6 + 4.334,700,0 \times z + \&c.$

Therefore $11.24x$ will be $(= 11.24 \times \overline{1.027,148 + z}$ $= 11.24 \times 1.027,148 + 11.24 \times z = 11.545,143,5 + 11.24z,$

and $21.76x^2$ will be $(= 21.76 \times \overline{1.055,033,0 + 2.054,296 \times z + \&c}$

$= 21.76 \times 1.055,033,0 + 21.76 \times 2.054,296 \times z + \&c)$

$= 22.957,518,0 + 44.701,480,9 \times z + \&c,$

and $11.54x^3$ will be $(= 11.54 \times \overline{1.083,675,0 + 3.165,099,0 \times z + \&c}$

$= 11.54 \times 1.083,675,0 + 11.54 \times 3.165,099,0 \times z + \&c)$

$= 12.505,609,5 + 36.525,242,4 \times z + \&c;$

and

and consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be = the compound quantity

$$\left\{\begin{array}{rrrrrr} 1.113,194,6 & + & 4.334,700,0 & \times z & + & \&c \\ - 12.505,609,5 & - & 36.525,242,4 & \times z & - & \&c \\ + 22.957,518,0 & + & 44.701,480,9 & \times z & + & \&c \\ - 11.545,143,5 & - & 11.24 & \times z & & \end{array}\right\}$$

$$= \left\{\begin{array}{rrrrrr} 24.070,712,6 & + & 49.036,180,9 & \times z & + & \&c \\ -24.050,753,0 & - & 47.765,242,4 & \times z & - & \&c \end{array}\right\}$$

$$= 0.019,959,6 + 1.270,938,5 \times z.$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $- 0.02$.

Therefore the compound quantity $0.019,959,6 + 1.270,938,5 \times z$ will alſo be $= 0.02$; and conſequently $1.270,938,5 \times z$ will be $(= 0.020,000,0 - 0.019,959,6) = 0.000,040,4$, and z will be $\left(= \frac{0.000,040,4}{1.270,938,5}\right) = 0.000,031,787$. Therefore x, or $1.027,148 + z$, will be $(= 1.027,148 + 0.000,031,787) = 1.027,179,787$; or the root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be, very nearly, $= 1.027,179,787$.

Q. E. I.

Of this number 1.027,179,787, I believe the firſt nine figures, 1.027,179,78, to be exact: but we may be confident that, (if no miſtakes have been made in the calculation,) at leaſt the firſt ſeven figures, 1.027,179, will be exact,

exact, and consequently that the difference of this value of x from it's true value is less than 0.000,001, or one millionth part of 1, and, *à fortiori*, less than one millionth part of the true value of x. We have therefore now found (though not without a good deal of labour,) the value of this first, or least, root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ to a considerable degree of exactness.

Art. 34. But this least value of x will not enable us to solve Colonel Titus's Problem. For, if x is $=$ 1.027,179,787, we shall have x^2 $(= \overline{1.027,179,787}^2)$ $=$ 1.055,098,314,821,365,369, and consequently lx^2, or $16x^2$, will be $= 16 \times 1.055,098,314,821,365,369 =$ 16.881,573,037,141,845,904, which is less than 17, or m; whereas in the foregoing Solution of Colonel Titus's Problem, it is shewn that lx^2 is greater than m, or 17, and that y is $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$, which fraction, if $16x^2$ is less than 17, (as it is when x is $=$ 1.027,179,787,) is a quantity that cannot exist. We must therefore inquire, whether the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ has not another root, or, perhaps, two other roots, besides 1.027,179,787, and whether one or both the said roots may not be such as to enable us to solve the foregoing Problem.

Of

Of the Second and Third Roots of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 35. To find whether the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ has, or has not, any other root greater than 1.027,179,787, which has been already found, let us ſuppoſe that it has ſuch a root, and inquire whether any falſe or impoſſible concluſion will follow from ſuch a ſuppoſition. For, if no ſuch falſe or impoſſible concluſion follows from it, but it ſhould lead us to an equation that is evidently poſſible, the ſuppoſition itſelf muſt be concluded to be likewiſe true.

To avoid confuſion in the reaſonings that will be grounded on this ſuppoſition, let this ſuppoſed greater root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ be denoted by the letter f, and the root 1.027,179,787 (which we have already found,) be denoted by the letter e.

Then we ſhall have $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = 0.02$, and $e^4 - 11.54e^3 + 21.76e^2 - 11.24e$ alſo $= 0.2$, and conſequently $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = e^4 - 11.54e^3 + 21.76e^2 - 11.24e$. Therefore (adding $11.54f^3 + 11.24f$ to both ſides,) we ſhall have $f^4 + 21.76f^2 = e^4 + 11.54f^3 - 11.54e^3 + 21.76e^2 + 11.24f - 11.24e$, and (ſubtracting $e^4 + 21.76e^2$

$11.76e^2$ from both ſides,) $f^4 - e^4 + 21.76f^2 - 21.76e^2 = 11.54f^3 - 11.54e^3 + 11.24f - 11.24e$, or $f^4 - e^4 + 21.76 \times \overline{f^2 - e^2} = 11.54 \times \overline{f^3 - e^3} + 11.24 \times \overline{f - e}$. Therefore $\frac{f^4 - e^4}{f - e} + 21.76 \times \frac{\overline{f^2 - e^2}}{f - e}$ will be $= 11.54 \times \frac{f^3 - e^3}{f - e} + 11.24 \times \frac{f - e}{f - e}$, and conſequently $f^3 + f^2e + fe^2 + e^3 + 21.76 \times \overline{f + e}$ will be $= 11.54 \times \overline{f^2 + fe + e^2} + 11.24$, or $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e$ will be $= 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$; which is a cubick equation involving only one unknown quantity, to wit, f, the letter e denoting 1.027,179,787, or the root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, which has been already found. We muſt therefore now inquire, whether this cubick equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$ is a poſſible equation.

Art. 36. In order to make this inquiry it will be neceſſary to ſubſtitute, inſtead of e, e^2, and e^3, in the terms of this equation their ſeveral values, to wit, 1.027,179,787, $\overline{1.027,179,787}^2$ and $\overline{1.027,179,787}^3$. But, as theſe values would lead us to very long numbers, and make theſe ſubſtitutions, and the ſubſequent operations relating to this equation, extreamly laborious, I ſhall ſubſtitute only the firſt four figures of the value of e, to wit, 1.027, inſtead of e on this occaſion. And then, (ſince it has been ſhewn above, in art. 30, that $\overline{1.027}^2$ is

Of the Second and Third Roots of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 35. To find whether the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ has, or has not, any other root greater than 1.027,179,787, which has been already found, let us ſuppoſe that it has ſuch a root, and inquire whether any falſe or impoſſible concluſion will follow from ſuch a ſuppoſition. For, if no ſuch falſe or impoſſible concluſion follows from it, but it ſhould lead us to an equation that is evidently poſſible, the ſuppoſition itſelf muſt be concluded to be likewiſe true.

To avoid confuſion in the reaſonings that will be grounded on this ſuppoſition, let this ſuppoſed greater root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ be denoted by the letter f, and the root 1.027,179,787 (which we have already found,) be denoted by the letter e.

Then we ſhall have $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = 0.02$, and $e^4 - 11.54e^3 + 21.76e^2 - 11.24e$ alſo $= 0.2$, and conſequently $f^4 - 11.54f^3 + 21.76f^2 - 11.24f = e^4 - 11.54e^3 + 21.76e^2 - 11.24e$. Therefore (adding $11.54f^3 + 11.24f$ to both ſides,) we ſhall have $f^4 + 21.76f^2 = e^4 + 11.54f^3 - 11.54e^3 + 21.76e^2 + 11.24f - 11.24e$, and (ſubtracting $e^4 +$ $21.76e^2$

$21.76e^2$ from both ſides,) $f^4 - e^4 + 21.76f^2 - 21.76e^2 = 11.54f^3 - 11.54e^3 + 11.24f - 11.24e$, or $f^4 - e^4 + 21.76 \times \overline{f^2 - e^2} = 11.54 \times \overline{f^3 - e^3} + 11.24 \times \overline{f - e}$. Therefore $\frac{f^4 - e^4}{f - e} + 21.76 \times \frac{\overline{f^2 - e^2}}{f - e}$ will be $= 11.54 \times \frac{f^3 - e^3}{f - e} + 11.24 \times \frac{f - e}{f - e}$, and conſequently $f^3 + f^2e + fe^2 + e^3 + 21.76 \times \overline{f + e}$ will be $= 11.54 \times \overline{f^2 + fe + e^2} + 11.24$, or $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e$ will be $= 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$; which is a cubick equation involving only one unknown quantity, to wit, f, the letter e denoting 1.027,179,787, or the root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, which has been already found. We muſt therefore now inquire, whether this cubick equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$ is a poſſible equation.

Art. 36. In order to make this inquiry it will be neceſſary to ſubſtitute, inſtead of e, e^2, and e^3, in the terms of this equation their ſeveral values, to wit, 1.027,179,787, $\overline{1.027,179,787}^2$ and $\overline{1.027,179,787}^3$. But, as theſe values would lead us to very long numbers, and make theſe ſubſtitutions, and the ſubſequent operations relating to this equation, extreamly laborious, I ſhall ſubſtitute only the firſt four figures of the value of e, to wit, 1.027, inſtead of e on this occaſion. And then, (ſince it has been ſhewn above, in art. 30, that $\overline{1.027}^2$ is

729) = 11.088,365,977, and $10.963,149f$ is (= 10.963,149 × 1.027) = 11.259,154,023, which is greater than 11.088,365,977. Therefore, in this caſe, the ſingle term $10.963,149f$ will be greater than the ſingle term $10.513f^2$; and therefore *à fortiori* the binomial quantity $10.963,149f + f^3$ will be greater than the ſingle term $10.513f^2$, and conſequently cannot be ſubtracted from it, and therefore the equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ (which ſuppoſes ſuch ſubtraction to take place,) is impoſſible.

But, while f increaſes gradually from being equal to 1.027 to a greater magnitude, the ſingle term $10.513f^2$ will receive greater increments than the binomial quantity $10.963,149f + f^3$, and will, firſt, become equal to it, and afterwards will exceed it. For the equation reſulting from the ſuppoſition of it's becoming equal to it, to wit, the equation $10.513f^2 = 10.963,149f + f^3$ may eaſily be ſhewn to be a poſſible equation. For it will be a poſſible equation if the quadratick equation $10.513f = 10.963,149 + ff$ (ariſing from the diviſion of all it's terms by f) is a poſſible equation: and this quadratick equation $10.513f = 10.963,149 + ff$, or $10.513f - ff = 10.963,149$, or $f \times \overline{10.513 - f} = 10.963,149$, is poſſible, becauſe 10.963,149 is leſs than the ſquare of half 10.513, or the ſquare of 5.256, which is greater than 25. Therefore the cubick equation $10.513f^2 = 10.963,149f + f^3$ is a poſſible equation.

And the magnitude of f at the inſtant at which $10.513f^2$ is exactly equal to $10.963,149f + f^3$, or

$10.513f$ is $= 10.963,149 + ff$, may be found by reſolving the quadratick equation $10.513f - ff = 10.963,149$; which may be done as follows.

Subtract both ſides of this equation from the ſquare of $\frac{10.513}{2}$, or 5.256, which is $= 27.625,536$; and we ſhall have $\frac{10.513}{2}\Big|^2 - 10.513f + ff$ $(= 27.625,536 - 10.963,149) = 16.662,387$. Therefore $\frac{10.513}{2} - f$, or $5.256 - f$, will be $= \sqrt{16.662,387} = 4.081$, and conſequently 5.256 will be $= 4.081 + f$, and f will be $= 5.256 - 4.081 = 1.175$. Alſo $f - 5.256$ will be $= 4.081$, and conſequently f will be $(= 4.081 + 5.256) = 9.337$. Therefore, when f is $= 1.75$, the trinomial quantity $10.513f^2 - 10.963,149 - f^3$ will be $= 0$, and afterwards, when f is become $= 9.337$, the ſaid trinomial quantity will be again $= 0$; and conſequently, while f increaſes from 1.75 to 9.337, the trinomial quantity will have increaſed from 0 to ſome certain quantity, and have decreaſed from that quantity to 0; and therefore, if the ſaid greateſt quantity to which the ſaid trinomial quantity will have increaſed is not leſs than $0.019,154,023$, the ſaid trinomial quantity will, during it's ſaid increaſe and decreaſe, or during the increaſe of f from 1.175 to 9.337, become twice equal to $0.019,154,023$, or the abſolute term of the cubick equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$, to wit, once when f is a little greater than 1.175, and a ſecond time when f is a little leſs than 9.337;

9.337; and consequently the said cubick equation will be a possible equation, and will have two roots, of which the lesser will be a little greater than 1.175, and the greater will be a little less than 9.337.

It only remains that we shew that the trinomial quantity $10.513f^2 - 10.963,149f - f^3$ will, in the course of it's increase and decrease while f increases from 1.175 to 9.337, become greater than 0.019,154,023. And this will appear by computing it's value at the instant that f becomes $= 2$. For, if f is $= 2$, we shall have $f^2 = 4$, and $f^3 = 8$, and $10.513 \times f^2$ $(= 10.513 \times 4) =$ 42.052, and $10.963,149f$ $(= 10.963,149 \times 2) =$ 21.926,298, and $10.963,149f + f^3$ $(= 21.926,298 + 8) =$ 29.926,298, and $10.513f^2 - 10.963,149f - f^3$ $(= 42.052 - 29.926,298) =$ 12.125,702; which is much greater than 0.019,154,023. It follows therefore that, during the increase of f from 1.175 to 2, the trinomial quantity $10.513f^2 - 10.963,149f - f^3$ will have increased from 0 to 12.125,702, and therefore must at some former instant of time, and when f was very little greater than 1 175, have been $=$ 0.019,154,023. And consequently the cubick equation $10.513f^2 - 10.963,149f - f^3 =$ 0.019,154,023 will be a possible equation, and will have two roots, of which the lesser will be a little greater than 1.175, and the greater will be a little less than 9.337. Q. E. I.

Art. 38. And hence it follows that the original biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ (from which the said cubick equation $10.513f^2$

$- 10.963,149f - f^3 = 0.019,154,023$ was derived) will have two other roots beſides the root 1.027,179,787, found above, and that each of the ſaid other roots will be greater than the ſaid firſt root, and that the leſſer of them will be a little greater than 1.175, and the greater of them will be a little leſs than 9.337. Theſe roots we will now proceed to inveſtigate,

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The Inveſtigation of the Middle Root of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 39. Since 1.175 is now known to be nearly equal to the middle root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, we will ſubſtitute it inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover how nearly the value of that compound quantity reſulting from ſuch ſubſtitution will approach to 0.02, or the abſolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Now, if x is $= 1.175$, we ſhall have x^2 $(= \overline{1.175}^2)$ $= 1.380,625$, and x^3 $(= \overline{1.175}^3)$ $= 1.622,234,375$, and x^4 $(= \overline{1.175}^4)$ $= 1.906,125,390,625$. Therefore $11.24x$ will be $(= 11.24 \times 1.175) = 13.207,00$, and $21.76x^2$ will be $(= 21.76 \times 1.380,625) = 30.042,400,00$, and $11.54x^3$ will be $(= 11.54 \times 1.622,234,375)$

=

$= 18.720,584,687,50$; and conſequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be ($= 1.906,125,390,625 - 18.720,584,687,50 + 30.042,400,00 - 13.207,00 = 31.948,525,390,625 - 31.927,584,687,50$) $= 0.020,940,703,125$. This number is very nearly equal to 0.02, or the abſolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$; and therefore 1.175 muſt be very nearly equal to this middle value of x in the ſaid equation.

But as this value of the ſaid compound quantity is greater than 0.02, or the abſolute term of the equation, it may perhaps be thought that the number 1.175 is greater, inſtead of being leſs, (as it has been declared to be,) than the true value of this middle root of the ſaid equation. And therefore, to aſcertain this point, we will now ſubſtitute 1.176 inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover whether the ſaid quantity will be increaſed, or diminiſhed, by ſuch ſmall increaſe of x, and thence to determine whether 1.75 is greater, or leſs, than the true value of x.

Now, if x is $= 1.176$, we ſhall have x^2 ($= \overline{1.176}^2$) $= 1.382,976$, and x^3 ($= \overline{1.176}^3$) $= 1.626,379,776$, and x^4 ($= \overline{1.176}^4$) $= 1.912,622,616,576$. Therefore $11.24x$ will be ($= 11.24 \times 1.176$) $= 13.218,24$, and $21.76x^2$ will be ($= 21.76 \times 1.382,976$) $= 30.093,547,76$, and $11.54x^3$ will be ($= 11.54 \times 1.626,379,776$) $= 18.768,422,615,04$; and conſequently the whole com-

pound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be (= 1 912,622,616,576 — 18.768,422,615,04 + 30.093,547,76 — 13.218,24 = 32 006,170,376,576 — 31.986,662,615,04) = 0.019,507,721,536; which is leſs than 0.020,940,703,125, or the former value of the ſaid compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, when x was = 1.175, and alſo leſs than 0.02, or the abſolute term of the equation under conſideration. Therefore, while x increaſes from 1.175 to 1.176, the ſaid compound quantity will decreaſe from 0.020,940,703,125 (which is greater than the abſolute term 0.02,) to 0.019,507,721,536 (which is leſs than the ſaid abſolute term,) and therefore will, at ſome intermediate inſtant of time, or when x was of ſome intermediate magnitude between 1.175 and 1.176, have been equal to the ſaid abſolute term; or, in other words, the true value of this root of the ſaid equation will be of an intermediate magnitude between 1.175 and 1.176. Q. E. D.

Art. 40. We are now in poſſeſſion of two values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, to wit, the numbers 0.020,940,703,125 and 0.019,507,721,536, correſponding to the two very near values of x, 1.175 and 1.176, between which the true value of x in the preſent biquadratick equation lies. We may therefore now obtain a much nearer value of x than either 1.175 or 1.176 by proceeding according to the differential method mentioned above in the Scholium in page 97. For, ſince 1.175, and the true value of x in the preſent equation, and 1.176, are three quantities, or values of x, very nearly equal to each other, and the

three values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ correſponding to the ſaid three contiguous quantities, or values of x, (or reſulting from the ſubſtitution of them in the ſaid compound quantity,) are 0.020,940,703,125, 0.020,000,000, and 0.019,507,721,536, we may ſuppoſe that the difference of the firſt and ſecond values of x will be to the differenee of the firſt and third values of x in nearly the ſame proportion as the difference of the values of the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ correſponding to the firſt and ſecond values of x to the difference of the values of the ſaid compound quantity correſponding to the firſt and third values of x; that is, that $x - 1.175$ will be to $1.176 - 1.175$ in nearly the ſame proportion as 0.020,940,703,125 — 0.020,000,000 is to 0.020,940,703,125 — 0.019,507,721,536, or that $x - 1.175$ will be to 0.001 in nearly the ſame proportion as 0.000,940,703,125 is to 0.001,432,981,589; whence it follows that $x - 1.175$ will be nearly $= \frac{0.001 \times 0.000,940,703,125}{0.001,432,981,589} = \frac{0.000,000,940,703,125}{0.001,432,981,589} =$ 0.000,656,46, and conſequently that x will be $(= 0.000,656,46 + 1.175) = 1.175,656,46$. Therefore this middle root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be $= 1.175,656,46$.

Q. E. I.

If no miſtake has been made in the foregoing calculations, this value of x will be exact in, at leaſt, the firſt ſix figures 1.175,65, and, probably, in the firſt ſeven figures 1.175,656.

The

The Application of this Second Value of x *in the Biquadratick Equation* $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, *or the Middle Root of the ſaid Equation, to the Solution of Colonel Titus's Problem.*

Art. 41. We will now apply this value of x to the ſolution of Colonel Titus's Problem.

Since x is $= 1.175,656$, and b, or the ſecond of the unknown numbers a, b, and c that are required to be found, is ſuppoſed to be $= xa$, we ſhall have $b = 1.175,656 \times a$.

Alſo we ſhall have x^2 $(= \overline{1.175,656}^2)$ $= 1.382,167,030,336$, and conſequently lx^2, or $16x^2$, $(= 16 \times 1.382,167,030,336) = 22.114,672,485,376$, and $lx^2 - m$ $(= 16x^2 - 17 = 22.114,672,485.376 - 17) = 5.114,672,485,376$, and $mx - l$ $(= 17x - 16 = 17 \times 1.175,656 - 16 = 19.986,152 - 16) = 3.986,152$, and conſequently $\frac{lx^2 - m}{mx - l} = \frac{5.114,672,485,376}{3.986,152} = 1.283,110,2$. Therefore y (which is $= \frac{lx^2 - m}{mx - l}$, or $\frac{16x^2 - 17}{17x - 16}$) will be $= 1.283,110,2$, or (neglecting the laſt figure) $1.283,110$; and c (which is $= ya$) will be $= 1.283,110 \times a$.

Therefore bc will be $= 1.175,656 \times a \times 1.283,110 \times a$ $(= 1.175,656 \times 1.283,110 \times aa = 1.508,495,970,160 \times aa$, and $aa + bc$ will be $= aa + 1.508,495,970,160 \times aa = 2.508,495,970,160 \times aa$.

Put $aa + bc$ is $= 16$.

Therefore

Therefore 2.508,495,970,160 × *aa* will alſo be = 16; and conſequently the ſquare-root of 2.508,495,970,160 × *aa* will be = the ſquare-root of 16, or to 4; that is, 1.583,823 × *a* will be = 4; and conſequently *a* will be $= \frac{4}{1.583,823} = 2.525,534.$

This value of *a* is exact in the firſt five figures 2.525,5, and ſomewhat too great in the ſixth figure 3, which ought to be a 1, the more accurate value of it, according to Dr. Wallis's computation of it, being 2.525,513,986,744,158.

Art. 42. Having thus found *a* to be = 2.525,534, we ſhall have *b* (= *xa* = 1.175,656 × *a* = 1.175,656 × 2.525,534) = 2.969,159, and *c* (= *ya* = 1.283,110 × *a* = 1.283,110 × 2.525,534) = 3.240,537; of which two numbers the former, to wit, 2.969,159, is exact in the firſt ſix figures 2.969,15, and the latter, to wit, 3.240,537, is exact in the firſt five figures 3.240,5, the more accurate values of *b* and *c* being, according to Dr. Wallis's computation of them, 2.969,152,708,619,848 and 3.240,580,681,617,174. Therefore the three unknown numbers *a*, *b*, and *c*, which Colonel Titus's Problem required to be found, are now diſcovered to be 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c. Q. E. I.

Art. 43. We have now found the firſt ſet of values of the unknown numbers *a*, *b*, and *c*, that will anſwer the conditions of Colonel Titus's Problem, which are 2.525,5 &c, 2.969,15 &c, and 3.240,5 &c. But there is alſo a ſecond ſet of numbers that will anſwer the conditions of the

the Problem. And theſe may be diſcovered by means of the third, or greateſt, root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, which we have already diſcovered to be nearly equal to, and ſomewhat leſs than, 9.337. This root we will therefore now proceed to inveſtigate to a greater degree of exactneſs.

The Inveſtigation of the Greateſt Root of the Biquadratick Equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Art. 44. In the firſt place we will ſubſtitute 9.337 inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover how near the value of the ſaid compound quantity reſulting from this ſubſtitution will approach to 0.02, or the abſolute term of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$.

Now, if x is $= 9.337$, we ſhall have x^2 $(= \overline{9.337}^2)$ $= 87.179,569$, and x^3 $(= 9.337^3) = 813.995,635,753$, and x^4 $(= \overline{9.337}^4) = 7600.277,251,025,761$, and $11.24x$ $(= 11.24 \times 9.337) = 104.947,88$, and $21.76x^2$ $(= 21.76 \times 87.179,569) = 1897.027,121,44$, and $11.54x^3$ $(= 11.54 \times 813.995,635,753) = 9393.509,636,589,62$. Therefore the whole compound quantity

x^4

$x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be ($= 7600.277,251,025,761 - 9393.509,636,589,62 + 1897.027,421,44 - 104.947,88$) $= 9497.304,672,465,761 - 9498.457,516,589,62$. This quantity is an impoſſible quantity, becauſe the latter quantity 9498.457,516,589,62 is greater than the former quantity 9497.304,672,465,761, from which it is to be ſubtracted. This unexpected reſult is owing to our having, in art. 36, ſubſtituted 1.027 inſtead of 1.027,179,787 for e in the terms of the equation $f^3 + ef^2 + e^2f + e^3 + 21.76f + 21.76e = 11.54f^2 + 11.54ef + 11.54e^2 + 11.24$; which was done for the ſake of abridging the calculations neceſſary to thoſe ſubſtitutions, but which made the final equation $10.513f^2 - 10.963,149f - f^3 = 0.019,154,023$ a little different from what it ſhould have been, and thereby gave us the two values of f derived from the ſuppoſition that the abſolute term of that equation was equal to 0, (and which were found in art. 37, by reſolving the quadratick equation $10.513f - ff = 10.963,149$, to be 1.175 and 9.337) a little different from what they ſhould have been, the former value of f in that equation, to wit, 1.175, being a little greater than it ſhould be, and the latter value of it, to wit, 9.337, being a little leſs than it ſhould be. We will therefore now ſuppoſe x, or the greateſt root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, of which we are now in ſearch, to be nearly equal to 9.339, and will ſubſtitute 9.339 inſtead of x in the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$, in order to diſcover how near the reſult of ſuch ſubſtitution will approach to the abſolute term 0.02 of the ſaid biquadratick equation.

Art. 45.

Art. 45. Now, if x is $= 9.339$, we ſhall have xx ($= \overline{9.339}^2$) $= 87.216,921$, and x^3 ($= \overline{9.339}^3$) $= 814.518,825,219$, and x^4 ($= \overline{9.339}^4$) $= 7606.791,308,720,241$, and $11.24x$ ($= 11.24 \times 9.339$) $= 104.970,36$, and $21.76x^2$ ($= 21.76 \times 87.216,921$) $= 1897.840,200,96$, and $11.54x^3$ ($= 11.54 \times 814.518,825,219$) $= 9399.547,243,027,26$, and $x^4 + 21.76x^2$ ($= 7606.791,308,720,241 + 1897.840,200,96$) $= 9504.631,509,680,241$, and $11.54x^3 + 11.24x$ ($= 9399.547,243,027,26 + 104.970,36$) $= 9504.517,603,027,26$. Therefore the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be $= 9504.631,509,680,241 - 9504.517,603,027,26 = 0.113,906,652,981$; which is more than five times 0.02, or the abſolute term of the propoſed equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. We therefore now know with certainty that, while x increaſes from 9.337 to 9.339, the binomial quantity $x^4 + 21.76x^2$ (which, when x is equal to 9.337, is leſs than the binomial quantity $11.54x^3 + 11.24x$) will have increaſed ſo as, firſt, to have become equal to the binomial quantity $11.54x^3 + 11.24x$, and afterwards to have become greater than the ſaid binomial quantity by the difference 0.113,906,652,981, which is greater than five times the abſolute term 0.02. And therefore it is certain that during the ſaid increaſe of x from 9.337 to 9.339, there muſt have been ſome one inſtant of time at which the exceſs of the binomial quantity $x^4 + 21.76x^2$ above the binomial quantity $11.54x^3 + 11.24x$ was equal to 0.02, or the abſolute term of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$; or, in

other

other words, there muſt be ſome value of x greater than 9.337, but leſs than 9.339, that will make the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ (which is the exceſs of the binomial quantity $x^4 + 21.76x^2$ above the binomial quantity $11.54x^3 + 11.24x$) exactly equal to 0.02, or will be a root of the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$. This root we will therefore now proceed to inveſtigate more exactly by Mr. Raphſon's method of approximation, and for that purpoſe will ſuppoſe x to be equal to $9.339 - z$, and will ſubſtitute $9.339 - z$ inſtead of it in the terms of the ſaid equation.

Art. 46. Now, if x is $= 9.339 - z$, we ſhall have

$xx\ (= \overline{9.339 - z}^2 = \overline{9.339}^2 - 2 \times 9.339 \times z +$ &c $= \overline{9.339}^2 - 18.678 \times z +$ &c$) = 87.216,921,$
$- 18.678 \times z +$ &c,

and $x^3\ (= \overline{9.339 - z}^3 = \overline{9.339}^3 - 3 \times \overline{9.339}^2 \times z +$ &c
$= \overline{9.339}^3 - 3 \times 87.216,921 \times z +$ &c
$= \overline{9.339}^3 - 261.650,763 \times z +$ &c)
$= 814.518,825,219 - 261.650,763 \times z +$ &c,

and $x^4\ (= \overline{9.339 - z}^4 = \overline{9.339}^4 - 4 \times \overline{9.339}^3 \times z +$ &c
$= \overline{9.339}^4 - 4 \times 814.518,825,219 \times z +$ &c
$= \overline{9.339}^4 - 3258.075,300,876 \times z +$ &c)
$= 7606.791,308,720,241 - 3258.075,300,$
$876 \times z +$ &c.

Therefore

Therefore $11.24x$ will be $(= 11.24 \times \overline{9.339 - z} = 11.24 \times 9.339 - 11.24 \times z) = 104.970,36 - 11.24z$; and $21.76x^2$ will be $(= 21.76 \times \overline{87.216,921 - 18.678 \times z + \&c} = 21.76 \times 87.216,921 - 21.76 \times 18.678 \times z + \&c) = 1897.840,200,96 - 406.433,28 \times z + \&c$, and $11.54x^3$ will be $(= 11.54 \times \overline{814.518,825,219 - 261.650,763 \times z + \&c} = 11.54 \times 814.518,825,219 - 11.54 \times 261.650,763 \times z + \&c) = 9399.547,243,027,26 - 3019.449,805,02 \times z + \&c$. And consequently the whole compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ will be equal to the compound quantity

$$\left\{\begin{array}{llll} 7606.791,308,720,241 & - & 3238.075,300,876 & \times z + \&c \\ - 9399.547,243,027,26 & + & 3019.449,805,02 & \times z - \&c \\ + 1897.840,200,96 & - & 406.433,28 & \times z + \&c \\ - 104.970,36 & + & 11.24 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{llll} 9504.631,509,680,241 & - & 3664.508,380,876 & \times z + \&c \\ -9504.517,603,027,26 & + & 3030.089,805,02 & \times z - \&c \end{array}\right\}$$

$$= 0.113,906,652,981 - 633.818,775,856 \times z \ \&c.$$

But the compound quantity $x^4 - 11.54x^3 + 21.76x^2 - 11.24x$ is $= 0.02$.

Therefore the compound quantity $0.113,906,652,981 - 633\ 818,775,856 \times z$ will alſo be $= 0.02$. And conſequently (adding $633.818,775,856 \times z$ to both ſides,) we ſhall have $0.113,906,652,981 = 0.02 +$

633.818,

633.818,775,856 × z, and (ſubtracting 0.02 from both ſides,) 0.093,906,652,981 = 633.818,775,856 × z, and (dividing both ſides by 633.818,775,856) we ſhall have $z\ (= \frac{0.093,906,652,981}{633.818,775,856}) = 0.000,148,1$. Therefore x, or $9.339 - z$, will be $(= 9.339,000,0 - 0.000,148,1) = 9.338,851,9$; that is, the third and greateſt root of the biquadratick equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$ will be $= 9.338,851,9$.

Q. E. I.

Art. 47. We will now apply this greateſt value of x to the ſolution of Colonel Titus's Problem.

Since x is $= 9.338,851,9$, and b, or the ſecond of the three unknown numbers a, b, and c that are required to be found, is $= xa$, we ſhall have $b = 9.338,851,9 \times a$.

Alſo we ſhall have $x^2\ (= \overline{9.338,851,9})\ = 87.214,154,810,133,61$, and conſequently $lx^2\ (= 16x^2 = 16 \times 87.214,154,810,133,61) = 1395.426,476,962,137,76$, and $lx^2 - m\ (= 16x^2 - 17 = 1395.426,476,962,137,76 - 17) = 1378.426,476,962,137,76$, and $mx\ (= 17x = 17 \times 9.338,851,9) = 158.760,482,3$, and $mx - l\ (= 17x - 16 = 158.760,482,3 - 16) = 142.760,482,3$, and conſequently $\frac{lx^2 - m}{mx - l} = \frac{1378.426,476,962,137,76}{142.760,482,3} = 9.655,518,4$. Therefore y

(which is $= \frac{ls^2 - m}{ms - l}$, or $\frac{16s^2 - 17}{17s - 16}$) will be = 9.655,518,4, and c (which is $= ya$) will be = 9.655,518,4 × a.

Therefore bc will be (= 9.338,851,9 × a × 9.655,518,4 × a = 9.338,851,9 × 9.655,518,4 × aa) = 90.171,456,355,324,96 × aa, and $aa + bc$ will be = aa + 90.171,456,355,324,96 × aa = 91.171,456,355,324,96 × aa.

But $aa + bc$ is = 16.

Therefore 91.171,456,355,324,96 × aa will be = 16; and conſequently the ſquare-root of 91.171,456,355,324,96 × aa will be = the ſquare-root of 16, or to 4; that is, 9.548,374,5 × a will be = 4; and conſequently

a will be = $\frac{4}{9.548,374,5}$ = 0.418,919,47.

This value of a is exact in all it's eight figures, in which it agrees with the value of it computed by Dr. Wallis to nine places of figures, which is 0.418,919,470. This is a very conſiderable degree of exactneſs.

Art. 48. Having thus found a to be = 0.418,919,47, we ſhall have b (= xa = 9.338,851,9 × a = 9.338,851,9 × 0.418,919,47) = 3.912,226,888, and c (= ya = 9.655,518,4 × a = 9.655,518,4 × 0.418,919,47) = 4.044,884,650; of which two numbers the former, to wit, 3.912,226,888, is exact in the firſt eight figures,

3.912,

3.912,226,8, and the latter, to wit, 4.044,884,650, is alſo exact in the firſt eight figures, 4.044,884,6; the more accurate values of b and c, according to Dr. Wallis's computation of them, being 3.912,226,866 and 4.044,884,670. Therefore the three unknown numbers a, b, and c, which Colonel Titus's Problem required to be found, are now diſcovered to be 0.418,919,47 &c, 3.912,226,8, &c, and 4.044,884,6, &c. Q. E. I.

This Problem is therefore now compleatly ſolved by Mr. Frend's method of proceeding, as it had been before by that of Dr. Wallis: and Mr. Frend's Solution has this advantage over that of Dr. Wallis, that it ſaves us the trouble of thoſe very tedious and perplexing Algebraick multiplications and diviſions which were neceſſary in Dr. Wallis's Solution, and which it is very difficult to perform without making ſome ſlip, or error, either in the ſigns + and − that are to be prefixed to the terms, or in the values of them.

OBSERVATIONS

ON

MR. RAPHSON's METHOD OF RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

Observations on Mr. Raphson's *Method of Reſolving Affected Equations of all Degrees by Approximation.*

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Article 1. IN the foregoing Tract I have given a pretty full explanation of Monſieur de Lagny's Method of Extracting the Roots of Numbers by Approximation, and I have likewiſe mentioned Mr. Raphſon's more ſimple and eaſy, though leſs exact, method of performing the ſame thing. But both theſe methods may be applied to the reſolution of all ſorts of equations, thoſe which are called *affected* equations*, or in which the unknown quantity occurs in more than one term,

* This expreſſion of *affected equations* ſeems to require ſome further explanation. It was introduced by the celebrated Vieta, the great improver of modern Algebra. He has many expreſſions peculiar to himſelf, and which have not been adopted by ſubſequent Algebräiſts. Amongſt theſe are the following ones: —He calls a ſet of quantities that are in continual geometrical

term, as well as thoſe which are called *pure* equations or in which the unknown quantity occurs in only one term, and which are reſolved by the mere extraction of the roots of given numbers. And in all affected equations beyond

proportion, (ſuch as the quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c,) a ſet of *ſcalar* quantities, or *magnitudines ſcalares*; and, when there are ſeveral of theſe *ſcalar* quantities connected with each other by the ſigns + and −, or by Addition and Subtraction, (as in the compound quantity $x^3 + ax^2 - b^2x^1$,) he calls the higheſt quantity, or that which is fartheſt in the ſcale of quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, (to wit, the quantity x^3 in the ſaid compound quantity $x^3 + ax^2 - b^2x^1$,) *the power* of the fundamental quantity x, or of the ſecond term in the ſaid ſcale; and he calls the lower ſcalar quantities, which are involved in the ſecond and third terms of the ſaid compound quantity $x^3 + ax^2 - b^2x^1$, to wit, the quantities x^2 and x^1, (or, in our preſent language, the inferiour powers of x,) ſcalar quantities of a *parodic* degree to x^3, or *the power* of the fundamental quantity x. This word *parodic* I take to be derived (though Vieta does not tell us ſo,) from the Greek words παρὰ and ὁδὸς, which ſignify *near* and *a way*, or *road*, becauſe theſe inferiour ſcalar quantities, x^2 and x^1, lie *in the way* as you paſs along in the ſcale of the aforeſaid quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, from 1 to x^3, which he calls *the power* of x in the ſaid compound quantity $x^3 + ax^2 - b^2x^1$. Theſe inferiour ſcalar quantities x^2 and x^1 are therefore *parodic*, or *ſituated in the way to*, or are *leading to*, the ſaid *power*, or higher ſcalar quantity, x^3. He then proceeds to define *a pure power* and *an affected power*, and tells us, that *a pure power* is a ſcalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* ſcalar quantity, and that *an affected power* is a ſcalar quantity

beyond biquadraticks, or thoſe of the fourth power, theſe methods of approximation are the only methods that can be taken for diſcovering their roots, or the values of the unknown quantities contained in them. And even in cubick

quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, ſcalar quantities, combined with co-efficients that raiſe them to the ſame dimenſion as the *power* itſelf, or make them *homogeneous* to it, and conſequently capable of being added to it, or ſubtracted from it. Thus x^5 alone is *a pure power* of x, namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is *an affected power* of x, namely, it's fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* ſcalar quantities, x^3 and x^4, which are multiplied into bb and a, in order to make them *homogeneous to*, or *of the ſame dimenſion with*, x^5 itſelf, and conſequently capable of being added to it, or ſubtracted from it. And he ſeems to have uſed the word *affected* in ſpeaking of ſuch a compound quantity as $x^5 + ax^4 - b^2x^3$, becauſe the magnitude of the higheſt power of x, to wit, x^5, was changed, or altered, or made greater, or leſs, than it would otherwiſe be, by the addition of the parodic quantity ax^4, and the ſubtraction of the parodic quantity b^2x^3; which increaſe, or diminution, or change, in the principal quantity x^5 he ſeemed to think might be well expreſſed by ſaying it was *affected* by it's connection with the ſaid parodic quantities. There are ſome expreſſions in his book that ſatisfy me that this was the idea he annexed to the word *affected*. See Schooten's edition of Vieta's Works, publiſhed at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expreſſions *a pure power* and *an affected power*, the meaning of the correſponding expreſſions

cubick and biquadratick equations, though particular methods have been invented by mathematicians, for the accurate reſolution of moſt of the caſes of theſe equations, (to wit, the rules called Cardan's Rules for the reſolution of moſt caſes of cubick equations, and the rules invented by Lewis Ferrari of Bologna in Italy, about the year 1545, and explained at large in Bombelli's Algebra, in the year 1579, and thoſe afterwards invented by M. Des Cartes, and publiſhed in his Geometry in the year 1637, for the reſolution of biquadratick equations, by the mediation of cubick equations,) it will be found that theſe methods of approximation will, for the moſt part,

preſſions of *a pure equation* and *an affected equation* follows from it of courſe; *a pure equation* ſignifying an equation in which a pure power of an unknown quantity is declared to be equal to ſome known quantity; ſuch as the equation $x^5 = 79$; and *an affected equation* ſignifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, ſome inferiour powers of the ſame unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the ſaid higheſt power of the ſaid unknown quantity,) is declared to be equal to ſome known quantity; ſuch as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expreſſion *an affected equation*. But, as the language of Vieta has not been adopted by ſubſequent writers of Algebra, I ſhould think it would be more convenient to call them by ſome other name. And, perhaps, thoſe of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$,

and

part, enable us to find the values of their roots to any propoſed degree of exactneſs, with leſs trouble than the particular and accurate methods above-mentioned, which have been invented for that purpoſe. So that theſe methods of reſolving equations by approximation ought to be conſidered as of the higheſt utility, and as being abſolutely neceſſary to the completion of the Doctrine of the Reſolution of Algebräick Equations, which is the moſt important branch of the Science of Algebra.

Art. 2. But it is not ſo eaſy to determine, which of theſe two methods of approximation, Mr. Raphſon's, or Mr. de Lagny's, deſerves to be preferred to the other on

and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, becauſe they would be equations in which a *binomial* quantity, or quantity conſiſting of two terms that involved the unknown quantity x, is declared to be equal to a known quantity; and, for a like reaſon, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I obſerve, in his excellent new Mathematical and Philoſophical Dictionary, juſt now publiſhed (Feb. 2, 1795,) calls them *compound* equations; which is likewiſe a very proper name for them, and leſs obſcure than that of *affected* equations. And Mr. Kerſey, in his excellent Treatiſe of Algebra, publiſhed in the year 1673, likewiſe calls them *compound* equations. But I ſhould think the former appellation of *binomial, trinomial, quadrinomial, quinquinomial,* and, in general, *multinomial* equations would be rather more deſcriptive of their nature.

theſe

theſe occaſions. Mr. Raphſon's is certainly much ſimpler than the other, becauſe it proceeds by conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as being only a ſimple equation, and reſolving it accordingly, or by the mere operation of Diviſion; whereas, in Mr. de Lagny's method, the ſaid new, or transformed, equation is conſidered as a quadratick equation, and reſolved accordingly; which, when a (or the firſt near value of the root, that is ſuppoſed to be already known,) is a number conſiſting of five, or ſix, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. Raphſon's method in reſolving all affected equations, more eſpecially when the number a conſiſts of more than two figures: but it muſt be confeſſed that the celebrated Dr. Halley (who had much experience, and was an excellent judge of theſe matters,) was of a different opinion, and gave the preference to Mr. de Lagny's method, which he has therefore taken the pains to explain in a better manner than had been done by Mr. de Lagny himſelf, and likewiſe to illuſtrate by examples, in his Tract in the Philoſophical Tranſactions, Number 210, intitled, "*A New, Exact, and Eaſy, Method of finding the "Roots of any Equations Generally, and that without any "previous Reduction,*" which was publiſhed in the year 1694. On the other hand we may obſerve, that Mr. Raphſon always continued to give his own method the preference, after the publication of the tracts of Mr. de Lagny and Dr. Halley upon the ſubject, as well as before their publication, when he tells us he had himſelf had the thought

thought of adopting the principle which was afterwards followed by Mr. de Lagny and Dr. Halley, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater eaſe and ſimplicity of the other method, in which the ſaid transformed equation is conſidered and treated as a ſimple equation. And Sir Iſaac Newton, in his method of reſolving equations by approximation, (which differs very little from Mr. Raphſon's,) ſeems alſo to prefer Mr. Raphſon's practice, of treating the transformed equation as a more ſimple equation, to that of Mr. de Lagny and Dr. Halley, of treating the ſaid equation as a quadratick equation. I therefore cannot but recommend it to all young Algebraiſts to ſtudy Mr. Raphſon's excellent Treatiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, with great attention, and to endeavour to make themſelves maſters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to ſay that they will thereby acquire more uſeful knowledge in Algebra, towards the buſineſs of reſolving affected, or compound, or multinomial, equations, than by reading all that has been written by Harriot and Des Cartes, and his learned Commentator Van Schooten, and all his other Commentators, and their numerous followers, on the boaſted doctrine of the Generation of Equations one from another, by ſuppoſing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and ſo on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and likewiſe

been the leaſt difficulty in underſtanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatim et litteratim*, as follows:

PROBLEMA XXIV.

Æquationum Quintæ Potestatis Adfectarum Solutio.

Proponatur $-aaaaa + 7aaaa - 20aaa + 155aa = 10{,}000.$

Hoc eſt, $-aaaaa + baaaa - caaa + daa = f.$

Theor. $x = \dfrac{f + ggggg + cggg - bgggg - dgg}{4bggg + 2dg - 5gggg - 3cgg}$

Sit $g = -5$

$f + ggggg + cggg - bgggg - dgg = -3875$
$4bggg + 2dg - 5gggg - 3cgg = -9675) (-3875{,}0) + {,}4 = x$

$-5{,}$
$+{,}4$

$g = -4{,}6$

$f + ggggg + cggg - bgggg - dgg = -420{,}36896$
$4bggg + 2dg - 5gggg - 3cgg = -7659{,}736) -420{,}36896$
$(+{,}055 = x$

$-4{,}6$
$+{,}055$

$g = -4{,}545$

$f +$

$f + gggggg + cggg - bgggg - dgg = -5,960359465465625$
$4bggg + 2dg - 5gggg - 3cgg = -7410,748) - 5,9603594$
$(+ ,00080428 = x$

$-4,545$
$+ ,000,80428$

$a = -4,544,195,72$ *

In this Problem the letter a is uſed for the unknown quantity, or root of the equation, which is uſually denoted by the letter x; and the letter g is uſed for the firſt near value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter a; and the letter x is uſed for the difference between g, the firſt near value of the root of the equation, and a, it's true value, which difference has been denoted in the two foregoing Tracts by the letter z. So that, if we expreſs the enun-

* To this ſolution I have, in my copy of Mr. Raphſon's Tract, ſubjoined the following Note:

Numerus 4.544,195,72 eſt radix æquationis $a^5 + 7a^4 + 20a^3 + 155a^2 = 10,000$; quod hic obſcurè innuitur ſub ſpecie radicis negativæ æquationis $-a^5 + 7a^4 - 20a^3 + 155a^2 = 10,000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ diſcendâ et excolendâ deterrentur, ex hiſce radicibus negativis et aliis quantitatibus negativis, ſeu (ut hodierni Algebræ ſcriptores abſurdè loquuntur,) *Nihilo minoribus*, ortum habent.

ciation of the foregoing Problem in the notation that has been uſed in the two foregoing Tracts, it will be as follows:

Proponatur $--xxxxx+7xxxx--20xxx+155xx=10{,}000$,

Sive $-x^5+7x^4-20x^3+155xx=10{,}000$,

Hoc eſt, $-xxxxx+bxxxx-cxxx+dxx=f$,

Sive $-x^5+bx^4-cx^3+dx^2=f$.

Theor. $z=\dfrac{f+aaaaa+caaa-baaaa-daa}{4baaa+2da-5aaaa-3caa}$,

or $z=\dfrac{f+a^5+ca^3-ba^4-da^2}{4ba^3+2da-5a^4-3ca^2}$.

Art. 4. Here, then, the equation propoſed by Mr. Raphſon to be reſolved, is ſaid to be $-x^5+7x^4-20x^3+155xx=10{,}000$, or $155xx-20x^3+7x^4-x^5=10{,}000$. But this is not the equation he reſolves; and, indeed, it is not a poſſible equation, becauſe the greateſt poſſible magnitude of the compound quantity $155xx-20x^3+7x^4-x^5$ is that which it has when the infinitely ſmall increment of the binomial quantity $20x^3+x^5$ becomes equal to the contemporary increment of the binomial quantity $155xx+7x^4$, that is, (if we put $\dot{x}$, or x with a point placed over it, for the infinitely ſmall increment of x,) when $20\times 3x^2\dot{x}+5x^4\dot{x}$ becomes equal to $155\times 2x\dot{x}+7\times 4x^3\dot{x}$, or when $60x^2+5x^4$ is $=310x+28x^3$, or when $60x+5x^3$ is $=310+28xx$, or

when $5x^3 - 28xx + 60x$ is $= 310$, or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by ſubſtituting 5.5 inſtead of x in the terms of the ſaid quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much leſs than 10,000, or the abſolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, and conſequently the ſaid equation is impoſſible. But Mr. Raphſon, though he ſets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, as the equation that is to be reſolved, yet really means to reſolve a quite different equation, to wit, the equation that reſults from ſuppoſing x to be a negative quantity, or from ſubſtituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the ſaid equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, inſtead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which ſubſtitution the ſaid equation will be converted into the equation $155 \times + xx - 20 \times - x^3 + 7 \times + x^4 - 1 \times - x^5 = 10,000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, which is evidently a poſſible equation, and of which the root is 4.544195,72, or the ſame number which he obtains by his ſolution of the Problem, and which, with the ſign $-$ prefixed to it, he calls the negative root of the propoſed equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$. Now all this perplexity would have been avoided, if Mr. Raph-

ſon had propoſed at firſt to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *poſitive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10{,}000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which equation is evidently poſſible, and can have only one root. And then all the ſteps of his ſolution would have been clear and eaſy, as will appear by reſolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$ according to the principles of his method; which may be done in the manner following:

The Reſolution of the Affected Equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, *by Mr. Raphſon's Method of Approximation.*

Art. 5. In conſidering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, it is, in the 1ſt place, eaſy to ſee that x muſt be greater than 1. For, if we ſuppoſe x to be $= 1$, we ſhall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than x.

In the ſecond place, if we ſuppoſe x to be $= 10$, we ſhall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10{,}000$, and

and $x^5 = 100{,}000$; ſo that x^4 alone will be equal to the abſolute term 10,000, and conſequently $x^5 + 7x^4 + 20x^3 + 155x$ muſt be very much greater than the ſaid abſolute term; and conſequently 10 muſt be much greater than x.

Thirdly, ſince x is leſs than 10 and greater than 1, let us ſuppoſe it to be equal to 5. Then we ſhall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875$) $= 13{,}875$; which is greater than the abſolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

We will therefore, in the 4th place, ſuppoſe x to be $= 4$. And then we ſhall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480$) $= 6576$; which is leſs than the abſolute term 10,000. Therefore 4 is leſs than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$ is greater than 4, but leſs than 5. And either of theſe values might very well ſerve for a firſt near value of the ſaid root, or for the baſis of a further approximation to it. Mr. Raphſon makes choice of 5, which is greater than the truth.

Art. 6. Let us then ſuppoſe a, or the firſt near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x. Then will x be $= a - z$, and conſequently xx will be $(= \overline{a - z}^2) = aa - 2az +$ &c, and x^3 will be $(= \overline{a - z}^3) = a^3 - 3a^2z +$ &c, and x^4 will be $(= \overline{a - z}^4) = a^4 - 4a^3z +$ &c, and x^5 will be $(= \overline{a - z}^5) = a^5 - 5a^4z +$ &c. Therefore $x^5 + 7x^4 + 20x^3 + 155xx$ will be $=$

$$\left\{\begin{array}{rrl} & & a^5 - 5a^4z + \&c, \\ + & 7 & \times \overline{a^4 - 4a^3z + \&c,} \\ + & 20 & \times \overline{a^3 - 3a^2z + \&c,} \\ + & 155 & \times \overline{aa - 2az + \&c,} \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} & & a^5 - 5a^4z + \&c, \\ + & 7a^4 & - 28a^3z + \&c, \\ + & 20a^3 & - 60a^2z + \&c, \\ + & 155aa & - 310az + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az +$ &c, will alſo be $= 10{,}000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) we ſhall have $a^5 + 7a^4 + 20a^3 + 155aa = 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (becauſe a is $= 5$, and conſequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13{,}875$, as has been ſhewn in art. 5,) we ſhall have $13{,}875 = 10{,}000 + 5a^4z + 28a^3z + 60a^2z +$

+

$+ 310az$, and conſequently (ſubtracting 10,000 from both ſides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times \overline{5a^4 + 28a^3 + 60a^2 + 310a}$. Therefore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a}$ $(= \frac{3875}{5 \times \overline{5}^4 + 28 \times \overline{5}^3 + 60 \times \overline{5}^2 + 310 \times 5} = \frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} = \frac{3875}{3125 + 3500 + 1500 + 1550} = \frac{3875}{9675}) = 0.4$. Therefore $a - z$, or x, will be $(= a - 0.4 = 5.0 - 0.4) = 4.6$; and 4.6 will be a ſecond near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

We muſt next try whether this ſecond near value of x is greater or leſs than it's true value; and for this purpoſe we muſt ſubſtitute it, inſtead of x, in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we ſuppoſe x to be $= 4.6$, we ſhall have xx $(= \overline{4.6}^2) = 21.16$, and x^3 $(= \overline{4.6}^3) = 97.336$, and x^4 $(= \overline{4.6}^4) = 447.7456$, and x^5 $(= \overline{4.6}^5) = 2059.62976$, and $155xx$ $(= 155 \times 21.16) = 3279.80$, and $20x^3$ $(= 20 \times 97.336) = 1946.720$, and $7x^4$ $(= 7 \times 447.7456) = 3134.2192$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 2059.62976 + 3134.2192 + 1946.720 + 3279.80) = 10{,}420.36896$; which is greater than 10,000,

or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art. 7. To find a third near value of the root of this equation, let a be ſuppoſed to be $= 4.6$, and z be the difference by which a, or 4.6, exceeds the true value of the ſaid root.

Then we ſhall have, as before, $x = a - z$, and conſequently $xx\ (= \overline{a - z}^2) = aa - 2az$ + &c, and $x^3\ (= \overline{a - z}^3) = a^3 - 3a^2z$ + &c, and $x^4\ (= \overline{a - z}^4) = a^4 - 4a^3z$ + &c, and $x^5\ (= \overline{a - z}^5) = a^5 - 5a^4z$ + &c, and $7x^4\ (= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z$ + &c, and $20x^3\ (= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z$ + &c, and $155xx\ (= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az$ + &c, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrrl} & a^5 & - 5a^4z & + \&c, \\ + & 7a^4 & - 28a^3z & + \&c, \\ + & 20a^3 & - 60a^2z & + \&c, \\ + & 155aa & - 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore

$$\left\{\begin{array}{rrrl} & a^5 & - 5a^4z & + \&c, \\ + & 7a^4 & - 28a^3z & + \&c, \\ + & 20a^3 & - 60a^2z & + \&c, \\ + & 155aa & - 310az & + \&c, \end{array}\right\}$$ will likewiſe be

$=$

$= 10{,}000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.6}^5 + 7 \times \overline{4.6}^4 + 20 \times \overline{4.6}^3 + 155 \times \overline{4.6}^2$, is $= 10{,}420.36896$.

Therefore $10{,}420.36896$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting $10{,}000$ from both sides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ $(= 5 \times \overline{4.6}^4 \times z + 28 \times \overline{4.6}^3 \times z + 60 \times \overline{4.6}^2 \times z + 310 \times 4.6 \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z) = 7659.7360 \times z$, and consequently z will be $\left(= \frac{420.36896}{7659.7360}\right)$ $= 0.0548$, or nearly 0.055. Therefore x, or $a - z$, or $4.6 - z$, will be nearly $(= 4.6 - 0.055,) = 4.545$; and consequently this number 4.545 will be a third near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

Now let this number 4.545 be substituted instead of xx in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to discover whether the result will be greater, or less,

leſs, than 10,000, or the abſolute term of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Now, if x be ſuppoſed to be $= 4.545$, we ſhall have $xx\ (= \overline{4.545}^2) = 20.657{,}025$, and $x^3\ (= \overline{4.545}^3) = 93.886{,}178{,}625$, and $x^4\ (= \overline{4.545}^4) = 426.712{,}681{,}850{,}625$, and $x^5\ (= \overline{4.545}^5) = 1939.409{,}139{,}011{,}090{,}625$, and conſequently $7x^4\ (= 7 \times 426.712{,}681{,}850{,}625) = 2986.988{,}772{,}954{,}375$, and $20x^3\ (= 20 \times 93.886{,}178{,}625) = 1877.723{,}572{,}500$, and $155xx\ (= 155 \times 20.657{,}025) = 3201.838{,}875$, and $x^5 + 7x^4 + 20x^3 + 155xx\ (= 1939.409{,}139{,}011{,}090{,}625 + 2986.988{,}772{,}954{,}375 + 1877.723{,}572{,}500 + 3201.838{,}875) = 10{,}005.960{,}359{,}465{,}465{,}625$; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. 8. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, let a be ſuppoſed to be $= 4.545$, and z be ſuppoſed to be the difference by which a, or 4.545, exceeds the true value of the ſaid root.

Then we ſhall, as before, have $x = a - z$, and conſequently $xx\ (= \overline{a - z}^2) = aa - 2az$ + &c, and $x^3\ (= \overline{a - z}^3) = a^3 - 3a^2z$ + &c, and $x^4\ (= \overline{a - z}^4) = a^4 - 4a^3z$ + &c, and $x^5\ (= \overline{a - z}^5) = a^5 - 5a^4z$ + &c,

and

and $7x^4 (= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z + \&c,$ and $20x^3 (= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z + \&c,$ and $155xx (= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az + \&c,$ and consequently $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrl} & a^5 - & 5a^4z + \&c, \\ + & 7a^4 - & 28a^3z + \&c, \\ + & 20a^3 - & 60a^2z + \&c, \\ + & 155a^2 - & 310az + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000.$

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z + \&c, - 28a^3z + \&c, - 60a^2z + \&c, - 310az + \&c,$ will likewise be $= 10{,}000;$ and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az.$

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa,$ or $\overline{4.545}^5 + 7 \times \overline{4.545}^4 + 20 \times \overline{4.545}^3 + 155 \times \overline{4.545}^2,$ is $= 10{,}005.960{,}359{,}465{,}465{,}625.$

Therefore $10{,}005.960{,}359{,}465{,}465{,}625$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az;$ and consequently (subtracting 10,000 from both sides,) $5.960{,}359{,}465{,}465{,}625$ will be $= 5a^4z + 28a^3z + 60a^2z +$

310az

$310az$ (= $5 \times \overline{4.545}^4 \times z + 28 \times \overline{4.545}^3 \times z + 60 \times \overline{4.545}^2 \times z + 310 \times 4.545 \times z = 5 \times 426.712,681,850,625 \times z + 28 \times 93.886,178,625 \times z + 60 \times 20\,657,025 \times z + 310 \times 4.545 \times z = 2133.563,409,253,125 \times z + 2628.813,001,500 \times z + 1239.421,500 \times z + 1408.950 \times z) = 7410.747,910,753,125 \times z$. Therefore z will be $(= \frac{5.960,350,465,465,625}{7410.747,910,753,125})$ = 0.000,804,28, and x, or $a - z$, or $4.545 - z$, will be (= 4.545,000,000 − 0.000,804,28) = 4.544,195,72. Therefore 4.544,195,72 will be a fourth near value of the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

This number 4.544,195,72, agrees with the number found by Mr. Raphſon, in all it's figures.

Art. 9. The foregoing reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, has been performed at great length, in order to ſet forth, in as clear a manner as poſſible, the ſeveral reaſonings upon which the arithmetical operations uſed in it are grounded, as well as the ſaid operations themſelves. And by ſo doing the ſubject is rendered ſo much eaſier than in Mr. Raphſon's very conciſe and compreſſed way of treating it, (in which all the reaſonings are dropped, and only the arithmetical operations are exhibited,) that, though the above reſolution of the ſaid equation is three, or four, times as long as Mr. Raphſon's, yet I am fully perſuaded that it may be read and underſtood in a third, or fourth, part of the time

time that is neceſſary to a thorough comprehenſion of Mr. Raphſon's reſolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$. But that this may appear the more clearly, I will now repeat the foregoing reſolution of this equation in the ſtyle and manner of Mr. Raphſon, by omitting the ſeveral reaſonings ſet forth in the foregoing articles, and making uſe of a Canon, or Theorem, for the purpoſe of computing the ſecond, third, and fourth values of z, in the ſame manner as Mr. Raphſon has done.

Art. 10. Since each of the three firſt ſucceſſive near values of x, or the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are ſucceſſively denoted by the letter a, is greater than the true value of x in the ſaid equation, or than the root of the ſaid equation, it follows that the ſecond, and third, and fourth near values of x will, each of them, be ſucceſſively denoted by the reſidual quantity $a - z$; and conſequently, by applying the reaſonings uſed in art. 6, in order to obtain the values of z, and of $a - z$, or x, we ſhall find that z will be, ſucceſſively, nearly equal to the value of the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$, and, therefore, that $a - z$, or x, will be, ſucceſſively, nearly equal to the value of the quantity a — the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem,

Theorem, or Canon, by the application of which we are to compute the firſt, and ſecond, and third near values of $a - z$, or the ſecond, and third, and fourth near values of x, after taking 5 for the firſt near value of it, or for the firſt value of a.

Now, if a is $= 5$, we ſhall have $z =$ the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4$. Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the ſecond near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Secondly, if a be $= 4.6$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360} =$ 0.0548, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Thirdly, if a be $= 4.545$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{5.960,359,465,465,625}{7410.747,910,753,125}$ $= 0.000,804,28$. Therefore $a - z$ will be $(= 4.545 - 0.000,804,28) = 4.544,195,72$; which will therefore be the fourth near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Q. E. I.

Art. 11.

Art. 11. Mr. Raphſon's Canon, or Theorem, for the value of z, is expreſſed more conciſely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uſes the letters b, c, d, and f, for the co-efficients 7, 20, and 155, of the fourth, third, and ſecond, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, and for 10,000, the abſolute term of that equation, reſpectively; which produces the following Canon, or Theorem, for the value of z, to wit, $z = \frac{a^5 + ba^4 + ca^3 + da^2 - f}{5a^4 + 4ba^3 + 3ca^2 + 2da}$. But it appears to me that, though we may ſeem to gain ſomething in point of brevity by uſing this very general notation, we loſe as much in the article of perſpicuity, which is a matter of much greater importance. However, this latter reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which is expreſſed in Mr. Raphſon's conciſe ſtyle and manner, and the foregoing more explicit reſolution of it in art. 5, 6, 7, and 8, (in which the reaſonings, on which the ſeveral arithmetical operations are grounded, are diſtinctly ſet forth and repeated,) are, both of them, the ſame in ſubſtance, and are, as I believe, the very beſt method that can be taken for diſcovering the root of the ſaid equation.

Of the Reſemblance of Mr. Raphſon's Method of Reſolving Numeral Equations by Approximation to Sir Iſaac Newton's Method of doing the ſame thing.

Art. 12. It has been obſerved above in art. 2, that Sir Iſaac Newton's method of reſolving numeral equations by

by approximation differed but little from Mr. Raphſon's, both methods being founded on the ſame principle of conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as a mere ſimple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than it's ſimple power; which reduces the reſolution of all equations, of whatever orders, to the reſolution of a ſimple equation, or, rather, to the reſolution of ſeveral ſucceſſive ſimple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Iſaac Newton's method and Mr. Raphſon's method perfectly agree; and, in finding the ſecond near value of x, or in making the firſt approximation to the true value of x, after having obtained, by conjecture, or trial, or in ſome other manner, the value of what has been here called a, or a firſt near value of x, or the root ſought, there is not the ſmalleſt difference between them. But in the inveſtigation of the third, and fourth, and other following near values of x, there is a little difference in their manner of proceeding, which the reader may be glad to ſee examined. I ſhall therefore now compare the two methods together, in the caſe of a very eaſy equation, by which Sir Iſaac Newton himſelf has thought proper to illuſtrate his method.

A COM-

A COMPARISON BETWEEN SIR ISAAC NEWTON'S AND MR. RAPHSON'S METHODS OF RESOLVING NUMERAL EQUATIONS BY APPROXIMATION.

Art. 13. Sir Isaac Newton's method of reſolving numeral equations by approximation, is explained by himſelf in his curious little Tract, intitled, *Analyſis per Æquationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Iſaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows:

Art. 14. Let it be required to reſolve the cubick equation $x^3 - 2x = 5$.

Here, in the firſt place, it is eaſy to ſee that x is ſomewhat greater than 2, but much leſs than 3. For, if x is taken equal to 2, we ſhall have $2x = 4$, and $x^3 = 8$, and conſequently $x^3 - 2x\ (= 8 - 4) = 4$; which is leſs than 5, or the true value of $x^3 - 2x$ in the propoſed equation: and, if x is taken equal to 3, we ſhall have $2x = 6$, and $x^3 = 27$, and conſequently $x^3 - 2x\ (= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the propoſed equation. Therefore the true value of x in that equation muſt be

much leſs than 3, and a little greater than 2. Let it therefore be ſuppoſed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be ſubſtituted, inſtead of x, in the propoſed equation $x^3 - 2x = 5$. This may be done as follows:

Since x is $= 2 + z$, we ſhall have $x^3 (= \overline{2}^3 + 3 \times \overline{2}^2 \times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times 2zz + z^3) = 8 + 12z + 6zz + z^3$, and $2x (= 2 \times \overline{2 + z}) = 4 + 2z$, and conſequently $x^3 - 2x (= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will alſo be $= 5$, and conſequently (ſubtracting 4 from both ſides,) $10z + 6zz + z^3$ will be $= 1$; and, (ſubtracting $6zz + z^3$ from both ſides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10} = \frac{1}{10} - \frac{6zz + z^3}{10} = 0.1 - \frac{6zz + z^3}{10}$, that is, z is leſs than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$. Therefore x, or $2 + z$, is leſs than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the ſaid quantity $\frac{6zz + z^3}{10}$; which, on account of the ſmallneſs of z, (which is leſs than $\frac{1}{10}$,) will be a very ſmall quantity in compariſon of z, or

of

of $\frac{1}{10}$, and, *à fortiori*, in compariſon of 2, and conſequently may be neglected. And therefore 2.1 will be a ſecond near value of x, or the root of the propoſed equation $x^3 - 2x = 5$, that will be a little greater than it's true value, but nearer to it than any other number that conſiſts of only two places of figures. Q. E. I.

This is the firſt ſtep of Sir Iſaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the aſſumption of the number 2, by conjecture and trial, for it's firſt near value. And in this firſt ſtep of the approximation Sir Iſaac Newton's and Mr. Raphſon's methods exactly co-incide.

Art. 15. But in the next ſtep of the approximation to the value x, in the ſaid equation $x^3 - 2x = 5$, the two methods are ſomewhat different from each other, though the number of new figures of the true value of x, that are exact in the next near values of it reſulting from both methods, is the ſame. The difference between the methods in this ſecond ſtage of the approximation is as follows:

Mr. Raphſon corrects the value of x, or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be ſomewhat greater than the truth,) by ſubtracting from it the unknown quantity by which it exceeds x, and which we may call v, and ſubſtituting $2.1 - v$ inſtead of x in the ſaid original equation, $x^3 - 2x = 5$, whereby it is transformed into

another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by reſolving the ſaid tranſformed equation as if it were only a ſimple equation, or by neglecting the terms which involve the ſquare and cube of v, on account of their ſmallneſs, juſt as we before neglected the terms $6zz$ and z^3 in the foregoing tranſformed equation $10z + 6zz + z^3 = 1$ for the ſame reaſon. But Sir Iſaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole proceſs of his approximation; but, inſtead of the ſaid original equation, he conſiders the former tranſformed equation, $10z + 6zz + z^3 = 1$, which was derived from it, and inveſtigates the value of it's root, z, to a greater degree of exactneſs than that to which it was before obtained. And this he does in the manner following:

Since it has been ſeen that z is leſs than 0.1, let the quantity by which 0.1 exceeds it be called v, ſo that z ſhall be $= 0.1 - v$; and let $0.1 - v$ be ſubſtituted, inſtead of z in the tranſformed equation $10z + 6zz + z^3 = 1$. This may be done as follows:

Since z is $= 0.1 - v$, we ſhall have

$zz \ (= \overline{0.1 - v}^2) = 0.01 - 0.2v + vv$,

and $z^3 \ (= \overline{0.1 - v}^3 = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv - v^3) = 0.001 - 0.03v + 0.3vv - v^3$,

and $10z \ (= 10 \times \overline{0.1 - v}) = 1 - 10v$,

and $6zz \ (= 6 \times \overline{0.01 - 0.2v + vv}) = 0.06 - 1.2v + 6vv$,

and conſequently

$$\left\{\begin{array}{l} 10z \\ +\,6zz \\ +\,z^3 \end{array}\right\} = \left\{\begin{array}{lll} 1.00 & - & 10.00v \\ +\,0.06 & - & 1.20v + 6vv \\ +\,0.001 & - & 0.03v + 0.3vv - v^3 \end{array}\right\}$$

$$= 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6zz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewiſe be $= 1$. And conſequently (adding $11.23v$ to both ſides,) we ſhall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (ſubtracting 1 from both ſides,) we ſhall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconſiderable in compariſon of 0.061 and $11.23v$) we ſhall have $0.061 = 11.23v$, or $11.23v = 0.061$; and conſequently (dividing both ſides by 11.23,) we ſhall have $v \left(= \frac{0.061}{11.23}\right) = 0.0054$. Therefore z, or $0.1 - v$, will be $(= 0.1 - 0.0054) = 0.0946$, and conſequently x, or $2 + z$, will be $(= 2 + 0.0946) = 2.0946$.

Q. E. I.

In this manner Sir Iſaac Newton finds the root of the propoſed equation $x^3 - 2x = 5$ to be equal to 2.0946, which is as near the truth as five figures can expreſs it.

Art. 16. He then carries the inveſtigation one ſtep further, by which he obtains the value of x exact to nine places of figures; and for this purpoſe he proceeds in the manner following:

The laſt transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, becauſe $6.3vv$ is greater than v^3. Since, therefore, v is greater than 0.0054, let us ſuppoſe it to be $= 0.0054 + w$; and let this binomial quantity be ſubſtituted, inſtead of v, in the transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation $11.23v - 6.3vv + v^3 = 0.061$, conſiſting of the ſame terms as the former, but in which the terms involving the unknown quantity v are all brought to the ſame ſide of the equation, and ranged according to the powers of v, beginning from it's loweſt power, or the ſimple power of v. This may be done in the manner following:

Since v is $= 0.0054 + w$, we ſhall have

$$vv\ (= \overline{0.0054+w}^2 = \overline{0.0054}^2 + 2 \times 0.0054 \times w + w^2)$$
$$= 0.000,029,16 + 0.0108 \times w + w^2,$$

and $v^3\ (= \overline{0.0054+w}^3 = \overline{0.0054}^3 + 3 \times \overline{0.0054}^2 \times w + 3 \times 0.0054 \times w^2 + w^3$
$$= 0.000,000,157,464 + 3 \times 0.000,029,16 \times w + 0.0162 \times w^2 + w^3)$$
$$= 0.000,000,157,464 + 0.000,087,48 \times w + 0.0162 \times w^2 + w^3,$$

and $11.23v\ (= 11.23 \times \overline{0.0054 + w}) = 0.060,642 + 11.23 \times w,$

and $6.3vv\ (= 6.3 \times \overline{0.000,029,16 + 0.0108 \times w + w^2})$
$$= 0.000,183,708 + 0.068,04 \times w + 6.3ww;$$

and

and consequently $11.23v - 6.3vv + v^3$ will be =

$$\left\{\begin{array}{l} 0.060,642 + 11.23 \times w \\ -0.000,183,708 - 0.068,04 \times w - 6.3 \times ww. \\ +0.000,000,157,464 + 0.000,087,48 \times w + 0.0162w^2 + w^3 \end{array}\right\}$$

$$= \left\{\begin{array}{l} 0.060,642,157,464 + 11.230,087,48 \times w + 0.0162w^2 + w^3 \\ -0.000,183,708 \quad - 0.068,04 \times w \quad - 6.3 \times w^2 \end{array}\right\}$$

$$= 0.060,458,449,464 + 11.162,047,48w - 6.2838w^2 + w^3.$$

But $11.23v - 6.3vv + v^3$ is $= 0.061$.

Therefore $0.060,458,449,464 + 11.162,047,48 \times w - 6.2838 \times ww + w^3$ will likewise be $= 0.061$; and consequently (subtracting 0.060,458,449,464 from both sides,) $11.162,047,48 \times w - 6.2838ww + w^3$ will be $(= 0.061,000,000,000 - 0.060,458,449,464) = 0.000,541,550,536$; and (neglecting the terms $6.2838ww$ and w^3, as inconsiderable in comparison of $11.162,047,48 \times w$,) we shall have $11.162,047,48 \times w = 0.000,541,550,536$, and consequently $w \left(= \frac{0.000,541,550,536}{11.162,047,48}\right) = 0.000,048,52$. Therefore v, or $0.0054 + w$, will be $(= 0.0054 + 0.000,048,52) = 0.005,448,52$, and z, or $0.1 - v$, will be $(= 0.100,000,00 - 0.005,448,58) = 0.094,551,48$, and x, or $2 + z$, will be $(= 2 + 0.094,551,48) = 2.094,551,48$; that is, the root of the proposed equation $x^3 - 2x = 5$ will be $= 2.094,551,48$. Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be shewn in a subsequent Article.

Art. 17. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6xx - x^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be tranſformed into another cubick equation, in which v will be the only unknown quantity; and then he determines the value of v by reſolving the ſaid tranſformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following:

Since x is $= 2.1 - v$, we ſhall have

$$xx \left(= \overline{2.1 - v}^2 = \overline{2.1}^2 - 2 \times 2.1 \times v + \&c\right)$$
$$= 4.41 - 4.2v + \&c,$$

and $x^3 \left(= \overline{2.1 - v}^3 = \overline{2.1}^3 - 3 \times \overline{2.1}^2 \times v + \&c\right.$
$$= 9.261 - 3 \times 4.41 \times v + \&c)$$
$$= 9.261 - 13.23 \times v + \&c,$$

and

and $2x\ (= 2 \times \overline{2.1 - v}) = 4.2 - 2v$,

and consequently $x^3 - 2x\ (= 9.261 - 13.23 \times v + \&c - 4.2 + 2v) = 5.061 - 11.23 \times v$ &c.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v$ &c, will likewise be $= 5$, and consequently (adding $11.23 \times v$ to both sides,) we shall have $5.061 = 5 + 11.23 \times v$, and (subtracting 5 from both sides,) we shall have $11.23 \times v = 0.061$, and consequently $v\ (= \frac{0.061}{11.23}) = 0.0054$. Therefore x, or $2.1 - v$, will be $(= 2.1 - 0.0054) = 2.0946$; or 2,0946 will be a third near value of the root of the proposed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very same with the third near value of it obtained above, in art. 15, by Sir Isaac Newton's method.

Art. 18. In this step of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the proposed equation $x^3 - 2x = 5$, the principal difference between the two methods seems to consist in this, to wit, that by Mr. Raphson's method we are obliged to raise the two first terms of the powers of the compound quantity $2.1 - v$, and consequently to raise the powers of the number 2 1, which consists of two figures; whereas in Sir Isaac Newton's method of proceeding, we had occasion only to raise the powers of

Art. 17. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6xx - x^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be transformed into another cubick equation, in which v will be the only unknown quantity; and then he determines the value of v by reſolving the ſaid tranſformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following:

Since x is $= 2.1 - v$, we ſhall have

$$xx \left(= \overline{2.1 - v}^2 = \overline{2.1}^2 - 2 \times 2.1 \times v + \&c\right)$$
$$= 4.41 - 4.2v + \&c,$$

and $x^3 \left(= \overline{2.1 - v}^3 = \overline{2.1}^3 - 3 \times \overline{2.1}^2 \times v + \&c\right.$
$$= 9.261 - 3 \times 4.41 \times v + \&c)$$
$$= 9.261 - 13.23 \times v + \&c,$$

and

and $2x\ (= 2 \times \overline{2.1 - v}) = 4.2 - 2v$,

and conſequently $x^3 - 2x\ (= 9.261 - 13.23 \times v + \&c - 4.2 + 2v) = 5.061 - 11.23 \times v$ &c.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v$ &c, will likewiſe be $= 5$, and conſequently (adding $11.23 \times v$ to both ſides,) we ſhall have $5.061 = 5 + 11.23 \times v$, and (ſubtracting 5 from both ſides,) we ſhall have $11.23 \times v = 0.061$, and conſequently $v\ (= \frac{0.061}{11.23}) = 0.0054$. Therefore x, or $2.1 - v$, will be $(= 2.1 - 0.0054) = 2.0946$; or 2,0946 will be a third near value of the root of the propoſed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very ſame with the third near value of it obtained above, in art. 15, by Sir Iſaac Newton's method.

Art. 18. In this ſtep of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the propoſed equation $x^3 - 2x = 5$, the principal difference between the two methods ſeems to conſiſt in this, to wit, that by Mr. Raphſon's method we are obliged to raiſe the two firſt terms of the powers of the compound quantity $2.1 - v$, and conſequently to raiſe the powers of the number 2 1, which conſiſts of two figures; whereas in Sir Iſaac Newton's method of proceeding, we had occaſion only to raiſe the powers of

the compound quantity $0.1 + v$, and conſequently to raiſe the powers of the number 0.1, which conſiſts of only one figure; which is ſomewhat eaſier than to raiſe the powers of 2.1. But both operations are ſo eaſy, that the difference of the labour of performing them is hardly worth conſidering. And, with reſpect to the ſimplicity of conception in the two methods, Mr. Raphſon's method ſeems to be preferable to Sir Iſaac Newton's, becauſe the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. 19. But in the next ſtep of the approximation by Mr. Raphſon's method, we ſhall find the labour of raiſing the powers of the value of x already found, to wit, the powers of 2.0946, to be conſiderably greater than that of raiſing the powers of the laſt preceeding ſupplement of it according to Sir Iſaac Newton's method, that ſupplement being only the decimal fraction 0.0054, in which there are only two ſignificant figures. This will appear by performing this ſtep of the approximation by Mr. Raphſon's method; which may be done as follows:

Art. 20. The laſt near value we found for x, or the root of the equation $x^3 - 2x = 5$, by Mr. Raphſon's method, was 2.0946. Now this near value of x is greater than it's true value. For, if we ſuppoſe x to be $= 2.0946$, we ſhall have $x^3 (= \overline{2.0946}^3) = 9.189,741,550,536$, and

$2x$

$2x$ ($= 2 \times 2.0946$) $= 4.1892$, and conſequently $x^3 - 2x$ ($= 9.189,741,550,536 - 4.1892$) $+ 5.000,541,550,536$; which is greater than 5, or the abſolute term of the equation $x^3 - 2x = 5$: and conſequently 2.0946 muſt be greater than the true value of the root of the ſaid equation.

We will therefore ſuppoſe x to be $= 2.0946 - w$, and ſubſtitute this reſidual quantity inſtead of x in the terms of the equation $x^3 - 2x = 5$.

Now, ſince x is $= 2.0946 - w$, we ſhall have

$$xx \left(= \overline{2.0946 - w}^2 = \overline{2.0946}^2 - 2 \times 2.0946 \times w + \&c\right)$$
$$= 4.387,349,16 - 4.1892 \times w + \&c,$$

and $x^3 \left(= \overline{2.0946 - w}^3 = \overline{2.0946}^3 - 3 \times \overline{2.0946}^2 \times w + \&c\right.$
$$= 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + \&c\left.\right)$$
$$= 9.189,741,550,536 - 13.162,047,48 \times w + \&c,$$

and $2x \left(= 2 \times \overline{2.0946 - w}\right) = 4.1892 - 2w$,

and conſequently $x^3 - 2x =$

$$\left\{\begin{array}{rrl} 9.189,741,550,536 & - & 13.162,047,48 \times w + \&c \\ - 4.189,2 & + & 2.000,000,00 \times w \end{array}\right\}$$
$$= 5.000,541,550,536 - 11.162,047,48 \times w + \&c.$$

But $x^3 - 2x$ is $= 5$.

Therefore $5.000,541,550,536 - 11.162,047,48 \times w + \&c$, will be $= 5$; and conſequently (adding 11.162,047,48

047,48 × w to both ſides,) we ſhall have 5.000,541,550,536 = 5 + 11.162,047,48 × w, and (ſubtracting 5 from both ſides,) 0.000,541,550,536 = 11.162,047,48 × w, or 11.162,047,48 × w = 0.000,541,550,536. Therefore w will be $(= \frac{0.000,541,550,536}{11.162,047,48})$ 0.000,048,52; and conſequently x, or 2.0946 — w, will be (= 2.094,600,00 — 0.000,048,52) = 2.094,551,48. Therefore 2.094,551,48 will be a fourth near value of x, or the root of the propoſed equation $x^3 - 2x = 5$.

Q. E. I.

This fourth near value of x is the very ſame with the fourth near value of it obtained above, in art. 16, by Sir Iſaac Newton's method.

Art. 21. In this laſt ſtage of Mr. Raphſon's approximation to the root of the propoſed equation $x^3 - 2x = 5$, we have been obliged to raiſe the powers of the number 2.0946, which conſiſts of five places of figures; whereas in Sir Iſaac Newton's way of proceeding we only raiſed the powers of the decimal fraction 0.0054, which contains only two ſignificant figures. But then in that way of proceeding we were obliged to multiply v, or 0.0054 + w, into 11.23, and vv, or the trinomial quantity 0.000,029,16 + 0.0108 × w + w^2, into 6.3; whereas in Mr. Raphſon's way of proceeding we have only to multiply x, or the binomial quantity 2.0946 — w, into the very ſimple co-efficient 2. So that, upon the whole, the difference of the labour of computation in the two methods is not very conſiderable, though it is rather

rather leſs in Sir Iſaac Newton's method than in Mr. Raphſon's. But in point of ſimplicity of conception Mr. Raphſon's method ſeems much ſuperiour to Sir Iſaac's, becauſe it never loſes ſight of the original equation $x^3 - 2x = 5$, which is to be reſolved.

And, further, we may obſerve, in favour of Mr. Raphſon's method, that it never requires us to raiſe any more than the two firſt terms of the binomial and reſidual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are ſubſtituted inſtead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Iſaac Newton's method it is neceſſary to raiſe the other terms of the binomial and reſidual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increaſes the number and intricacy of the operations of the inveſtigation. And therefore, upon the whole, I conſider Mr. Raphſon's method of approximating to the values of the roots of ſuch equations as preferable to Sir Iſaac Newton's.

A Proof of the Exactneſs of the Number 2.094,551,48, *that has been found by the foregoing Methods of Approximation for the Root of the Equation* $x^3 - 2x = 5$.

Art. 22. It remains that we prove the work to have been rightly performed, or that we ſhew that the laſt number

number 2.094,551,48, obtained by both theſe methods, is a very near value of the root x of the propoſed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plaineſt and beſt method of doing this is to ſubſtitute the number 2.094,551,48, inſtead of x, in the compound quantity $x^3 - 2x$, in order to diſcover whether the quantity reſulting from this ſubſtitution will be greater, or leſs, than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$: and, if it ſhall appear that the ſaid reſult is greater than 5, we may conclude that the ſaid number 2.094,551,48 is greater than the true value of x in the ſaid equation; and, if it ſhall appear that the ſaid reſult is leſs than 5, we may conclude that the ſaid number is leſs than the true value of x. And, when this has been thus diſcovered, we muſt, in the next place, endeavour to determine to how many figures this number 2.094,551,48 co-incides with the more accurate value of x: and, for this purpoſe, we muſt, if this number be leſs than x, increaſe it by the addition of an unit in the laſt place of figures; and, if it be greater than x, we muſt diminiſh it by the ſame ſmall quantity, and then ſubſtitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, inſtead of x, in the compound quantity $x^3 - 2x$. And, if it ſhall appear that the value of that compound quantity reſulting from ſuch ſubſtitution is greater, or leſs, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater,

greater, or leſs, than the true value of x, in the equation $x^3 - 2x = 5$, and conſequently that the ſaid true value is of an intermediate magnitude between 2.094,551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094,551,48$, we ſhall have

$xx = 4.387,145,902,370,190,4,$

and $x^3 = 9.189,102,942,785,417,810,201,792,$

and $2x = 4.189,102,96,$

and conſequently

$x^3 - 2x = 4.999,999,982,785,417,810,201,792;$

which number is ſomewhat leſs than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$. Therefore 2.094,551,48 muſt be ſomewhat leſs than the true value of x in the ſaid equation.

Secondly, ſince x is greater than 2.094,551,48, we muſt now compare it with 2.094,551,49, by ſubſtituting that number inſtead of it in the compound quantity $x^3 - 2x$.

Now, if x is taken $= 2.094,551,49$, or $2.094,551,48 + 0.000,000,01$, we ſhall have $x^3 \; (= \overline{2.094,551,48}^3 + 3 \times \overline{2.094,551,48}^2 \times \overline{0.000,000,01} + 3 \times 2.094,551,48 \times \overline{0.000,000,01}^2 + \overline{0.000,000,01}^3$

=

= 9.189,102,942, &c, + 3 × 4.387,145,902, &c × 0.000,000,01 + 0.000,000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 13.161,437,706, &c × 0.000,000,01 + 0.000,000,000 &c + 0.000,000,000, &c = 9.189,102,942, &c + 0.000,000,131, &c + 0.000,000,000, &c + 0.000,000,000, &c) = 9.189,103,073, &c; and $2x$ (= 2 × 2.094,551,49) = 4.189,102,98; and consequently $x^3 - 2x$ (= 9.189,103,07, &c − 4.189,102,98) = 5.000,000,09, &c; which is greater than 5. Therefore 2.094,551,49 must be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been shewn that 2.094,551,48 is less than the said true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and consequently all the figures of the number 2.094,551,48, which we found by the foregoing processes of Sir Isaac Newton's and Mr. Raphson's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of

Of the Difficulty of finding a, or the First Near Value of the Root of an Affected Equation, in certain Cases.

Art. 23. There is another difficulty that occurs sometimes in resolving high equations by approximation, whether by Sir Isaac Newton's method or by Mr. Raphson's; which indeed are substantially the same. The difficulty I mean, is that of finding the first near value of the root sought (which we have called a in this discourse,) to one, or two places of figures, in order to make it the basis of a further approximation to the true value of the root by either of these methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for all other roots are not worth considering,) this difficulty will not be great; because it will always be easy to find a tolerably near value of the root by conjectures and trials, and particularly by supposing x, or the root of the proposed equation, first to be equal to 1, and 2dly, to be $= 10$, and 3dly, to be equal to some short intermediate number between 1 and 10, consisting of only one figure, or, if the root appears to be greater than 10, by supposing it to be equal to 100, or 1000, and afterwards supposing it to be equal to some short intermediate number consisting of two figures; as was done above in art. 5, in finding the first near value of x in the equation $x^5 + 7x^4 + 20x^3 +$

= 9.189,102,942, &c, + 3 × 4.387,145,902, &c × 0.000,000,01 + 0.000,000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 13.161,437,706, &c × 0.000,000,01 + 0.000,000,000 &c + 0.000,000,000, &c = 9.189,102,942, &c + 0.000,000,131, &c + 0.000,000,000, &c + 0.000,000,000, &c) = 9.189,103,073, &c; and $2x$ (= 2 × 2.094,551,49) = 4.189,102,98; and consequently $x^3 - 2x$ (= 9.189,103,07, &c − 4.189,102,98) = 5.000,000,09, &c; which is greater than 5. Therefore 2.094,551,49 must be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been shewn that 2.094,551,48 is less than the said true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and consequently all the figures of the number 2.094,551,48, which we found by the foregoing processes of Sir Isaac Newton's and Mr. Raphson's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of

Of the Difficulty of finding a, or the First Near Value of the Root of an Affected Equation, in certain Cases.

Art. 23. There is another difficulty that occurs sometimes in resolving high equations by approximation, whether by Sir Isaac Newton's method or by Mr. Raphson's; which indeed are substantially the same. The difficulty I mean, is that of finding the first near value of the root sought (which we have called a in this discourse,) to one, or two places of figures, in order to make it the basis of a further approximation to the true value of the root by either of these methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for all other roots are not worth considering,) this difficulty will not be great; because it will always be easy to find a tolerably near value of the root by conjectures and trials, and particularly by supposing x, or the root of the proposed equation, first to be equal to 1, and 2dly, to be $= 10$, and 3dly, to be equal to some short intermediate number between 1 and 10, consisting of only one figure, or, if the root appears to be greater than 10, by supposing it to be equal to 100, or 1000, and afterwards supposing it to be equal to some short intermediate number consisting of two figures; as was done above in art. 5, in finding the first near value of x in the equation $x^5 + 7x^4 + 20x^3 +$

$155xx = 10{,}000$. But, when the equation conſiſts of terms connected together partly by the ſign +, and partly by the ſign −, and conſequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforeſaid method of conjectures and trials (though by no means uſeleſs,) is leſs expeditious and ſatisfactory in aſſiſting us to find the firſt near value of one of the roots than in the former caſe; and we are often puzzled to know which of the roots it would be moſt expedient to begin to inveſtigate. Now, in moſt of theſe caſes, I believe, it will be adviſeable to begin by inveſtigating the leaſt root, and for that purpoſe to expunge from the equation all the terms that have the ſign − prefixed to them, and to find, to about two places, or, at moſt, to three places, of figures, the root of the remaining equation. For this root will always be leſs than the leaſt root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be leſs than the only root of the original equation, if (notwithſtanding the appearances to the contrary,) it really has but one root. When the root of this ſecond, or curtailed, equation, has been diſcovered, it may be called a, and made the ground-work of an approximation to the leaſt root of the original equation, and the binomial quantity $a + z$ may be ſubſtituted in the original equation inſtead of x, and the transformed equation thence ariſing may be reſolved as if it was a mere ſimple equation, agreeably to Mr. Raphſon's method of approximation; and the value of z thereby obtained, being added to a, will give us a known

value

value of $a + z$, or a ſecond near value of the leaſt, or the only, root of the propoſed equation: after which we may proceed to find the ſaid leaſt, or only, root of the propoſed equation by a further proſecution of Mr. Raphſon's method of approximation above-deſcribed, untill we have determined it to the degree of exactneſs that we think neceſſary. This method of finding a firſt near value, a, of the leaſt root of a propoſed equation that ſeems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Diſcourſe on the Reverſion of Infinite Serieſes publiſhed in that Volume, in pages 724, 725, 726, 727, &c, - - - to page 761; which pages will be reprinted in the ſubſequent part of this preſent Collection of Tracts. And, with this improvement of it in the caſe of equations that have, or ſeem to have, more than one real and poſitive root, I believe it may ſafely be affirmed that Mr. Raphſon's Method of Reſolving Affected Equations is the beſt general method of effecting that purpoſe in all equations above quadraticks that has hitherto been diſcovered.

End of the Obſervations on Mr. Raphſon's *Method of Reſolving Affected Equations by Approximation.*

An EXPLICATION *of* SIMON STEVIN's General Rule, *to Extract One Root out of any Poſſible Equation in Numbers, either Exactly, or very nearly True.*

By JOHN KERSEY.

BEING THE TENTH CHAPTER OF THE SECOND BOOK OF MR. KERSEY'S ELEMENTS OF ALGEBRA.

Article 1. EQUATIONS falling under any of the forms in the fourteenth and fifteenth chapters of the firſt book of theſe Elements, are capable (as hath there been ſhewn) of perfect reſolutions in numbers; viz. the value of the root or roots ſought in any of thoſe equations may be found out and expreſſed exactly, either by ſome rational or irrational number or numbers; but the perfect reſolution of all manner of compound equations in numbers, I have not found in any author: and ſince an expoſition of the general method of Vieta, the rules of Huddenius and others to that purpoſe, would make a large treatiſe, and after all leave the curious Analyſt diſſatisfied, I ſhall not clogg theſe Elements with

a tedious diſcourſe upon thoſe difficult rules, (which at the beſt are exceeding tedious in operation, and ſome of them uncertain too,) but rather purſue my firſt deſign, which was to explain fundamentals, and ſuch rules as are certain and moſt important in this profound art. However, I ſhall lead the induſtrious learner a few ſteps farther in order to his underſtanding the reſolution of all manner of compound equations in numbers, and in this Chapter ſhall explain Simon Stevin's General Rule, which, with the help of the rules in the following Eleventh Chapter, will diſcover all the roots of any poſſible equation in numbers, either exactly, if they be rational, or very nearly true, if irrational.

QUESTION I.

If - - - $aaa + 26a$ is $= 40188$, what is the number a?

RESOLUTION.

This equation not falling under any of the three forms in Sect. I. chap. xv. book 1. cannot be reſolved by any of the canons in that chapter, and therefore according to Simon Stevin's general method I ſearch out the number a by trials, thus, viz.

1. I ſuppoſe - - - - $a = 1$
Thence it follows that - - $aaa = 1$
And - - - - $26a = 26$
Therefore - - $aaa + 26a = 27$;

Which 27 ought to have been 40188, but it's too little; whereby I find that, by ſuppoſing a to be 1, I did not hit upon the true number a; and therefore I make another trial, in like manner as before, viz.

2. I ſuppoſe - - - - $a = 10$
Thence it follows that - - $aaa = 1000$
And - - - - $26a = 260$
Therefore - - $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third trial, viz.

3. I ſuppoſe - - - $a = 100$
Thence it follows, that $aaa + 26a = 1,002,600$

Which 1,002,600 exceeds the juſt reſult, or abſolute number 40,188 in the latter part of the equation firſt propoſed, and therefore the true number a is leſs than 100; but the ſecond trial ſhews it to be greater than 10, and therefore the whole number which expreſſeth the exact, or, at leaſt, part of the value of a, muſt neceſſarily conſiſt of two characters, and conſequently the firſt (towards the left hand) muſt be one of theſe nine, 1, 2, 3, 4, 5, 6, 7, 8, 9. But, becauſe, by the ſecond inquiry,

10 was found too little, I now make trial with 2 for the firſt figure of the root a, viz.

4. I ſuppoſe - - - - $a = 20$
Thence - - $aaa + 26a = 8520$

Which reſult 8520 being yet leſs than the juſt reſult 40,188, I make trial again, viz.

5. I ſuppoſe - - - $a = 30$
Thence - - $aaa + 26a = 27780$

Which is yet too little; therefore,

6. I ſuppoſe - - - $a = 40$
Thence - - $aaa + 26a = 65,040$

Which 65,040 being greater than 40,188, it ſhews me that the true root or value of a is leſs than 40; but by the fifth trial it is greater than 30, and conſequently the firſt figure of the root is 3.

Now the ſecond character of the root muſt neceſſarily be one of theſe, viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and becauſe it hath been diſcovered that the true value of the root a is greater than 30, the ſecond character cannot be 0: I therefore make trial with 1, and ſuppoſe $a = 31$, which proving too little, I make trial with 32, 33, 34, &c. ſeverally, in like manner as before, and at length I find 34 to be the true number a ſought, by which the equation propoſed may be expounded; for, if $a = 34$, then conſequently $aaa + 26a = 40,188$.

Art. 2.

Art. 2. But if after trials made (as before) the value of *a*, the root ſought, happens to fall between two whole numbers that differ by unity; then trials are to be made with the leſſer whole number increaſed with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of *a* in ſome mixt number conſiſting of a whole number and ſome certain tenth parts of an unit: But if the ſaid value of *a* happens not to be expreſſed exactly by the ſaid leſſer whole number increaſed with certain tenth parts, then you are to make trials with the ſaid leſſer whole number increaſed with a decimal fraction having for it's numerator a number greater than 10, but leſs than 100; and for it's denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the root *a* when it's fractional part is exactly equal to ſome decimal fraction, or elſe approach infinitely near to the ſaid exact value when it is irrational or ſurd, as in this following:

QUESTION II.

If - - - $aaaa + 50a$ is $= 184{,}638.6801$; (or, $184{,}638\frac{6801}{10000}$;) what is the number *a*?

RESOLUTION.

First, I ſuppoſe $a = 1$; but this proving too little, I put $a = 10$; this alſo proving too little, I aſſume $a = 100$, which after trial I find to be greater than the true number *a*, and conſequently the number *a* falls between 10 and

and 100; then, making trial with 20, I find it too little; but making trial with 30, I find this too great; and therefore the true root a falls between 20 and 30. Again, making trial with 21, I find it too great: but 20 was before found too little; therefore the true root a is between 20 and 21; then I make trial with 20.1, (that is, $20\frac{1}{10}$;) 20.2; 20.3, &c. and at length find 20.7 to be the true number a ſought; for if $a = 20.7$, (that is, $20\frac{7}{10}$,) it will make $aaaa + 50a = 184{,}638.6801$ the equation propoſed *.

But if 20.7 had proved too little, and 20.8 too great, then trials muſt have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, &c. In like manner, if 20.7 had been too little, but 20.71 (that is, $20\frac{71}{100}$) too great, then trials muſt have been made with 20 701, (that is, $20\frac{701}{1000}$;) 20.702; 20.703, &c. This will be partly exemplified in reſolving the equation in this following—

QUESTION III.

If - - - $aaa + 20aa = 1954$, what is the number a?

Anſ. - - $a = 8.308$, &c. found out by trials, as before.

* This method of proceeding is very convenient for finding the two firſt figures of the value of a; but then it will be expedient (for the ſake of greater diſpatch,) to have recourſe to Mr. Raphſon's method of approximation, which will uſually double, or nearly double, the number of figures in the value of a at every operation. See his *Analyſis Æquationum Univerſalis.*

Art. 3.

Art. 3. When the value of (*a*) the required root of an equation happens to be leſs than unity, then trial is to made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now ſuppoſe .1 (that is, $\frac{1}{10}$) be too great, but .01 (that is, $\frac{1}{100}$) too little; then trial muſt be made with .02 | .03 | .04 | &c. until you have found out the greateſt figure that muſt ſtand in the ſecond place of the decimal fraction expreſſing the root ſought; ſuppoſing then ſuch figure to be found 8, viz. that .08 (or $\frac{8}{100}$) is leſs, but .09 (or $\frac{9}{100}$) is greater than the root, trial muſt be made with .081, (that is, $\frac{81}{1000}$,) .082 | .083 | &c. as in this following—

QUESTION IV.

If - - $aaa + 3240a$ is $= 269$, what is the number a?

Anſw. - a is $= .083$, &c. that is, $\frac{83}{1000}$, &c.

Art. 4. The preceeding examples may ſuffice to ſhew the uſe of this general method when all the terms of the unknown part of an equation are affirmative, (viz. when + is prefixed to each term,) in which caſe there is but one affirmative root; in the ſearch whereof by trials (as before) if the numbers aſſumed ſeverally for the value of the root ſought do aſcend, or become greater and greater, then the abſolute numbers reſulting from thoſe aſſumed values will likewiſe aſcend; and contrary, if the aſſumed roots do deſcend from a greater to a leſs, the reſults will likewiſe grow leſs and leſs: whence, by comparing an abſolute number reſulting from an aſſumed root with the juſt abſolute number of the equation propoſed, you may certainly

certainly know (if the ſaid reſult and juſt abſolute be not equal to one another) whether you are to take a number greater or leſs than that laſt before aſſumed.

But when the unknown part of an equation conſiſts of affirmative and negative terms mingled one with another, then the ſearch by trials will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed root muſt be taken greater or leſs than that which was taken next before. Moreover, a compound equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative roots as there be unities in the index of the higheſt unknown power, viz. a cubical equation may be ſo conſtituted that it ſhall have three different affirmative roots, a biquadratick equation four ſeveral roots; and ſo of higher equations, as will be ſhewn in the following Chap. 11. But, in what manner ſoever any poſſible equation is conſtituted in rational numbers, this general method will always find out one affirmative root, either exactly true, or at leaſt very near the truth; as will farther appear by the following queſtions:

QUESTION V.

If - - - - $aaa - 22aa + 157a = 360$, what is the number a?

RESOLUTION.

1. I ſuppoſe - - - - - $a = 1$

Thence it follows that $aaa - 22aa + 157a = 136$

Which 136 is leſs than the juſt abſolute number 360, and therefore I make another trial, viz.

2. I ſuppoſe - - - - $a = 10$
Thence it follows that $aaa - 22aa + 157a = 370$

Which 370 exceeds the juſt abſolute number 360, and therefore I conclude there is one affirmative value of a (either rational or irrational) between 1 and 10, which value, after trials made with 2, 3, 4, 5, I find to be 5: this will conſtitute the equation propoſed; for if $a = 5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other roots or values of a, to wit, 8 and 9, each of which will likewiſe conſtitute the equation firſt propoſed; but how they are found out will be ſhewn in Sect. 9. of the following Chap. 11.

QUESTION VI.

If - - - - $3200a - aaa = 46,577$ (juſt,) what is the number a?

RESOLUTION.

1. I ſuppoſe - - $a = 1$
Thence - $3200a - aaa = 3199$ (leſs than juſt.)

2. I ſuppoſe - - $a = 10$
Thence - $3200a - aaa = 31000$ (leſs than juſt.)

3. I

3. I ſuppoſe - - $a = 100$
Thence - $3200a - aaa = -680{,}000$ (leſs than juſt.)

Now becauſe the ſecond reſult (or abſolute number) + 31000 is affirmative, and the laſt reſult — 680,000 is negative, I make trials with numbers between 10 and 100 for the value of a; for, if the equation propoſed be poſſible, before the affirmative reſults fall off to negatives, there will be a root, or value of a, producing an affirmative reſult either exactly equal, or very near to the juſt reſult 46,577; therefore,

4. I ſuppoſe - - $a = 20$
Thence - $3200a - aaa = 56{,}000$ (greater than juſt.)

Now becauſe by taking 20 for the value of a, the reſult 56,000 exceeds the juſt reſult 46,577; but by taking 10 for a, the reſult 31000 happened to be leſs than the ſaid 46577; it ſhews that there is one affirmative root or value of a between 10 and 20, which root, after trials made with intermediate numbers (as in former examples) will be found 15.7, &c. Moreover, becauſe, by ſuppoſing $a = 20$, the reſult 56,000 happened to exceed the juſt reſult 46,577, but, by putting $a = 100$, the reſult — 680000 proved to be leſs than the ſame 46,577, it ſhews there is an affirmative value of a between 20 and 100, which value after trials made will be found 47: ſo that there are two affirmative roots or values of a found out, to wit, 15.7, &c. (or $15\frac{7}{10}$, &c.) and 47; the former of which will nearly, and the latter exactly, conſtitute the equation propoſed.

Art. 5.

Art. 5. Florimond de Beaune in the latter of two ſmall treatiſes printed in 1659, concerning the nature, conſtitution and limits of equations *, ſhews how to find-out limits within which the roots of all compound equations not aſcending above the biquadratick kind are confined; which limits, when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the trials in the general method before delivered: As, in the laſt Example, where

The equation propoſed was $3200a - aaa = 46,577$

Firſt, becauſe aaa muſt be ſubtracted from $3200a$ and leave a remainder equal to 46,577, it preſuppoſeth	- - aaa to be $\sqsupset 3200a$
Therefore, by dividing each part by a, - -	- - - $aa \sqsupset 3200$
And by extracting the ſquare-root out of each part, it follows that - - -	- - - $a \sqsupset 56.5$, &c.
Again, from the equation propoſed, by tranſpoſition it is evident that - - -	$3200a - 46,577 = aaa$
Whence it is alſo manifeſt	- - $3200a \sqsubset 46,577$
And conſequently, by dividing each part by 3200,	- - - $a \sqsubset 14.5$, &c.

* This tract of M. de Beaune is very clear and very uſeful. It is printed with Mr. Schooten's Comment on Des Cartes's Geometry.

Thus it is found that the value of a the root ſought is greater than 14.5, &c. but leſs than 56.5, &c. and therefore trials according to the general method aforeſaid need not be made with any numbers that are not within thoſe limits.

From the premiſes it is evident, that this general method finds not a perfect root of an equation, unleſs ſuch root be a whole number, or elſe a fraction exactly equal to ſome decimal fraction; or laſtly, a mixed number compoſed of a whole number and a perfect decimal fraction.

Note. When the co-efficients or known numbers multiplied into any of the unknown powers under the higheſt, (which muſt have no co-efficient but unity,) are vulgar (not decimal) fractions, or mixt numbers whoſe fractional parts are vulgar fractions; likewiſe, when the abſolute number that ſolely poſſeſſeth the latter part of the equation propoſed is a vulgar fraction, or a mixt number whoſe fractional part is a vulgar fraction; all thoſe vulgar fractions muſt be reduced to decimal fractions, or elſe the equation muſt be reduced to another equation in integers (by Sect. 7. in the following Chap. 11.) before you enter upon the reſolution by trials as aforeſaid.

End of the Tenth Chapter of the Second Book of Mr. Kerſey's *Elements of Algebra.*

A RE-

A REMARK

ON

AN ERROR IN MONSIEUR CLAIRAUT'S ELEMENTS OF ALGEBRA.

By FRANCIS MASERES, Esq.
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A REMARK *on an Error in the Reaſoning of the late learned French Mathematician,* Monſieur CLAIRAUT, *in that Part of his* ELEMENTS OF ALGEBRA *in which he endeavours to prove the Rules of Multiplication laid down by Writers on Algebra concerning Negative Quantities.*

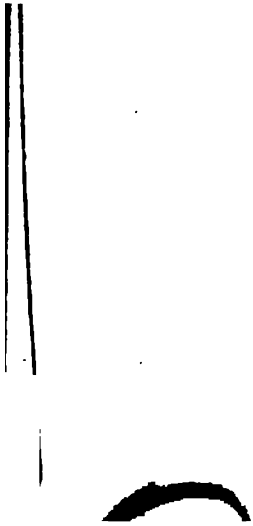
THERE is one writer of Algebra who has treated the ſubject with uncommon care and elegance, and, throughout the firſt part of his work, with great perſpicuity likewiſe. I mean Monſieur Clairaut of the French Academy of Sciences, in his *Elémens d'Algèbre* publiſhed at Paris in the year 1749. But even this writer has been betrayed into a moſt remarkable inſtance of falſe reaſoning, by his deſire of explaining to his readers that which in it's nature is not capable of being explained or underſtood, to wit, the propoſition "*that, if a negative quantity be multiplied by a negative quantity, the product will be an affirmative quantity.*" This propoſition he has attempted to demonſtrate in the 60th Section, or Article, of his book, page 73, in the following manner: He had already in Art. iii, page 4, explained the ſign — to ſignify the ſubtraction of the quantity to which it is prefixed from the quantity which immediately preceeds it, in theſe

words: " *Prenant le caractere* —, *qui ſe prononce* moins, *pour faire reſſouvenir que la quantité qu'il précéde doit être retranchée de celle qu'il ſuit.*" And in Articles xlii, xliii, xliv, xlv, xlvi, pages 48, 49, 50, 51, 52, 53, 54, he explains the manner of multiplying compound quantities, (which he calls *des quantités complexes ou polynomes, ou quantités composées de pluſieurs termes*,) by each other, and exemplifies it by the multiplication of the compound quantity $2a^3c^3 - 5a^4b + 6a^5$ by the compound quantity $3ab^2 - 4bcd$. And thus far he reaſons juſtly and clearly, uſing the ſign — only in the ſenſe of his firſt definition of it in Art. iii, page 4, as the ſign of the ſubtraction of the quantity to which it is prefixed from that which goes before it, in which firſt part of his book he ſeems to ſuppoſe all along that the quantities to be ſubtracted are leſs than thoſe from which they are to be ſubtracted. But then, when he has thus ſhewn clearly that, in the multiplication of compound quantities by compound quantities, the rule holds good, that — into — gives +, or that thoſe members of the product of the two compound quantities which ariſe from the multiplication of any two members of the compound quantities themſelves that are both marked with the ſign —, will be marked with the ſign +, he endeavours, in Art. lx, page 73, to extend this concluſion to the product of two independent, or ſeparate, negative quantities, or quantities which are marked with the ſign — without being preceeded by any other quantities from which they are ſubtracted. This lxth Article is in theſe words:

Pour nous aſſurer que la multiplication de — par — doit toûjours donner + au produit, voyons quelle lumiére nous pouvons

pouvons tirer de la méthode générale des multiplications donnée dans l'Art. xlv. Suivant cette méthode on voit très clairement que le produit d'une quantité telle que a — b *par une autre* c — d *doit être* ac — bc — ad + bd; *et on voit par conſéquent en même temps que le terme* bd, *qui eſt venu par la multiplication de* b *et de* d, *a le ſigne* +, *tandis que ſes produiſants* b *et* d *ont le ſigne* —. *Il ne reſte donc plus qu'à ſçavoir ſi, lorſque deux quantités negatives, telles que* b *et* d, *ne ſeront précedées d'aucune quantité poſitive, leur produit ſera encore* + bd. *Or c'eſt ce dont il eſt facile de reconnoitre la vérité, puiſque la méthode par laquelle on a découvert que le produit de* a — b *par* c — d *étoit* ac — bc — ad + bd, *ne ſpécifiant aucune grandeur particuliére ni à* a *ni à* c, *doit avoir encore lieu lorſque ces quantités ſont égales à zéro. Or, en ce cas, le produit* ac — bc — ad + bd *ſe reduit à* + bd. *Donc* — b × — *d eſt* = + bd.

Now in theſe words there is an obvious fallacy. For, though *a* and *c* are not particular numbers, or quantities which have a fixt relation to *b* and *d*, but they may be of many different magnitudes with reſpect to *b* and *d*, yet they muſt always be ſuppoſed to be *greater* than *b* and *d* reſpectively. For otherwiſe it will not be poſſible to ſubtract *b* and *d* from them, ſo as to produce the quantities *a* — *b* and *c* — *d*; the author not having hitherto given us any other idea of the ſign — but that of it's denoting the ſubtraction of the quantities to which it is prefixed from thoſe which go before them; in order to which it is neceſſary, and is conſtantly ſuppoſed in all the foregoing part of the book, that the ſaid quantities, to which the ſaid ſign is prefixed, ſhould be leſs than thoſe that go



A GENERAL METHOD

OF

INVESTIGATING THE TWO, OR THREE, FIRST FIGURES OF THE LEAST ROOT OF AN EQUATION THAT HAS MORE THAN ONE REAL AND AFFIRMATIVE ROOT.

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A General Method of inveſtigating the Two, or Three, firſt Figures of the leaſt Root of an Equation that has more than One real and affirmative Root.

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Article 1. IN the foregoing Appendix to Dr. Halley's Tract on the Reſolution of High Equations by Approximation, I have endeavoured to illuſtrate his method by a very full and diſtinct reſolution of the three equations which he has choſen for examples of it, and to compare it with Mr. Raphſon's method of reſolving the like equations, (which alſo proceeds by repeated approximations to the true values of the roots ſought,) by applying the latter method to the reſolution of the ſame equations which had before been reſolved by Dr. Halley's method; and the reſult of the compariſon was, that Mr. Raphſon's method appeared to me, for the moſt part, more convenient than Dr. Halley's. But in the application of both theſe methods of reſolving theſe equations a difficulty may ſometimes occur which it will be expedient to endeavour to remove. For, when the terms of an equation have many changes of their ſigns from + to — and from — to +, it may, perhaps, have many different roots, (I mean, real and affirmative roots,) of very dif-

ferent magnitudes from each other, and we may find ourſelves at a loſs to determine which of theſe ſeveral roots we ought to begin by, or go firſt in purſuit of. This difficulty, indeed, will not always occur in equations of this kind, becauſe it will ſometimes happen that the conditions of the problem from which the equation is derived, will point-out to us ſome limits to that root of the equation which is neceſſary to the ſolution of the problem, and which therefore ought to be the object of our inveſtigation; and in theſe caſes it is evident we muſt make uſe of theſe limits as our guides in our firſt conjectural approximation towards the value of the root that lies between, or near, them. But, when the conditions of the problems do not afford us ſuch aſſiſtance, we may be a good deal puzzled to know which of the ſeveral roots which the equation poſſibly may have, we ſhould endeavour to inveſtigate firſt. And in theſe caſes, I believe, it will often be found convenient to begin with the inveſtigation of *the leaſt* of the ſeveral different roots of the equation. The method of doing this, and the reaſons for chuſing to begin with the leaſt root in theſe caſes in preference to any of the other roots, will be the ſubject of the preſent diſcourſe.

Art. 2. In many equations of all degrees, it is eaſy to perceive that they can have but one root, that is, but one real and affirmative root. For, as to negative roots, they are in truth the real and affirmative roots of other equations conſiſting of the ſame terms, or members, as the equations of which they are ſaid to be the negative roots, but differently connected with each other by the

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ſigns + and —; or by addition and ſubtraction: and, as to impoſſible roots, they are mere imaginary quantities, of which the mind can form no idea; and they are called roots of the equations they belong to, only becauſe, if they are ſubſtituted in the ſaid equations inſtead of x, they will make the left-hand ſides of the equations be equal, or, rather, ſeem to be equal, to their ſeveral abſolute terms reſpectively; or, in other words, becauſe, if they are ſquared, and cubed, and raiſed to the fourth, or fifth, or other higher, powers, and the ſaid ſquares, and cubes, and other powers of them, are multiplied into the coefficients of the ſame powers of x in the equations to which they belong, they will make the left-hand ſides of ſuch equations be equal to their ſeveral abſolute terms reſpectively. But it is impoſſible to ſquare, or cube, or multiply, a quantity that cannot exiſt, or a non-entity; and therefore all that is ſaid about theſe impoſſible roots is little better than ſtark nonſenſe, and tends only to darken and diſgrace the Science of Algebra. Nor can I conceive the ſmalleſt reaſon for ever mentioning theſe roots, or indeed negative roots, in books of Algebra, unleſs it be to ſupport the truth of a favourite poſition concerning equations that has been laid down by modern writers of Algebra, to wit, "that every Algebraick equation has as many roots as there are units in the index of the higheſt powers of x contained in it;" which poſition would, without the admiſſion of negative and impoſſible roots, be moſt eminently falſe. For many equations of all degrees have in truth only one root, or quantity, really exiſting, and of which we can form a clear conception, that, being ſubſtituted inſtead of x in the terms of the left-hand ſide of the equation, will make

the reſult of them be equal to the abſolute term of the equation. And it is often eaſy to perceive that theſe equations can have but one ſuch root, though their terms may involve in them the cube, or the fourth power, or the fifth power, or any higher powers, of the unknown quantity x; of which I will mention two, or three, of the moſt remarkable inſtances.

Art. 3. In the firſt place, when all the terms on the left-hand ſide of the equation, or that involve in them any powers of the unknown quantity x, are added to each other, or connected together by the ſign $+$, it is obvious that ſuch an equation can have but one root.

Thus, for example, if the equation be $x^3 + 36xx + 44x = 1$, it is evident that x can have but one value. For, if x were to increaſe from it's firſt value, or the value which it has when the compound quantity $x^3 + 36x^2 + 44x$ becomes firſt equal to the abſolute term 1, to any greater magnitude, each of the three quantities x^3, $36x^2$, and $44x$ would, at the ſame time, increaſe continually, or without ever decreaſing, from it's firſt value to a greater quantity, and conſequently their ſum $x^3 + 36x^2 + 44x$ would alſo increaſe continually, or without ever decreaſing, from it's firſt value, or 1, to a greater quantity, and therefore could never become a ſecond time equal to 1.

And, in like manner, the biquadratick equation $x^4 + 80x^3 + 1998x^2 + 14{,}937x = 5000$ can have but one root. For, if x were to increaſe from it's firſt value, or the

the value it has when the compound quantity $x^4 + 80x^3 + 1998x^2 + 14{,}937x$ is firſt equal to 5000, to any greater quantity, each of the four terms x^4, $80x^3$, $1998x^2$, and $14{,}937x$ would increaſe at the ſame time continually, or without ever decreaſing, from it's firſt value to ſome greater quantity; and conſequently their ſum, or the compound quantity $x^4 + 80x^3 + 1998x^2 + 14{,}937x$, would alſo at the ſame time increaſe continually, or without ever decreaſing, and therefore could never become a ſecond time equal to the abſolute term 5000.

And the like would be evident in any other equation whatſoever, in which all the terms on the left-hand ſide of the equation, or all the terms that involve in them the powers of x, were added to each other, or connected together by the ſign $+$. Therefore every ſuch equation can have but one root.

Art. 4. Secondly, when the term that involves the higheſt power of x is greater than the ſum of all the other terms on the left-hand ſide of the equation, and the exceſs of the ſaid term above the ſaid ſum of all the other terms on that ſide of the equation is equal to the abſolute term of the equation, (as, for example, if x^3 is greater than $36xx + 44x$, and $x^3 - 36x^2 - 44x$ is equal to 1; or, if x^4 is greater than $80x^3 + 1998xx + 14{,}937x$, and $x^4 - 80x^3 - 1998xx - 14{,}937x$ is equal to 5000;) the equation can have but one root. And this root will be greater than the value of x that would reſult from a ſuppoſition that the abſolute term of the equation was equal to 0.

Thus,

Thus, for example, the root of the cub
$x^3 - 36x^2 - 44x = 1$ muſt be great
root of the equation $x^3 - 36x^2 - 44x =$
the root of the quadratick equation $xx -$
$= 0$, or $xx - 36x = 44$; and the roo
quadratick equation $x^4 - 80x^3 - 1998x^2$
$= 5000$ will be greater than the root of
$x^4 - 80x^3 - 1998xx - 14{,}937x = 0$,
root of the cubick equation $x^3 - 80xx -$
$14{,}937 = 0$, or $x^3 - 80xx - 1998x =$

Art. 5. The truth of theſe poſitions will
from conſidering the manner in which the
involving the powers of x in any of the
will increaſe while x increaſes from 0 *ad infi*

For, when x is very ſmall, all the other t
left-hand ſide of the equation, which will in
x^4, &c, will be much ſmaller than x, the ſai
x being continued proportionals to 1 and x.
be taken of ſo ſmall a magnitude that the pro
to x^2 ſhall be greater than any ratio of major
have been aſſigned. But, as x increaſes, the
(which involve x^2, x^3, x^4, &c,) will increaſe f
and the ratio of x to x^2 will decreaſe contin
having been a very great ratio of majority, til
a ratio of equality when x is equal to 1.
time all the powers of x, to wit, x^2, x^3, x^4,
be equal to each other and to 1. And, when
further from 1 to a greater magnitude, x^4 wi
than x, and likewiſe than x^2 and x^3, and it'

in any given portion of time will be greater than the contemporary increment of x, and likewiſe than the contemporary increments of either of the other powers of x that are lower than itſelf, to wit, x^2 and x^3. And, by this means, x^4, from having been leſs than either $14{,}937\,x$, or $1998\,x\,x$, or $80\,x^3$, will gain ground upon them, and, at ſome one point of time, become equal to the ſum of all the three together; and at this time the compound quantity $x^4 - 80x^3 - 1998x^2 - 14{,}937x$ will be equal to 0, and the value of x will be the root of the biquadratick equation $x^4 - 80x^3 - 1998x^2 - 14{,}937x = 0$, or the root of the cubick equation $x^3 - 80x^2 - 1998x - 14{,}937 = 0$, or $x^3 - 80x^2 - 1998x = 14{,}937$. And, when x increaſes further from this value *ad infinitum*, the increment of x^4 (which has already been greater than the ſum of the contemporary increments of $80x^3$, and $1998x^2$, and $14{,}937x$, ſo as to enable x^4, from having been at firſt leſs than either of the three quantities, to become equal to all the three put together,) will continue to be greater than the ſum of the three contemporary increments of thoſe three quantities, $80x^3$, $1998x^2$, and $14{,}937x$, and in a continually-increaſing ratio of majority. And conſequently the compound quantity $x^4 - 80x^3 - 1998x^2 - 14{,}937x$, or the exceſs of the ſingle quantity x^4 above the ſum of the three quantities $80x^3$, $1998x^2$, and $14{,}937x$, will increaſe continually, or without ever decreaſing, from 0 *ad infinitum*, while x increaſes from being equal to the root of the cubick equation $x^3 - 80x^2 - 1998x = 14{,}937$ *ad infinitum*. Therefore the ſaid compound quantity $x^4 - 80x^3 - 1998x^2 - 14{,}937x$ will, at

ſome one inſtant or time during it's ſaid increaſe from 0 *ad infinitum*, become equal to the abſolute term 5000, or to any other given quantity whatſoever; but it will be equal to ſuch quantity only once during it's ſaid increaſe from 0 *ad infinitum*. And conſequently the equation $x^4 - 80x^3 - 1998x^2 - 14{,}937x = 5000$ can have but one root, and that root muſt be greater than the root of the biquadratick equation $x^4 - 80x^3 - 1998x^2 - 14{,}937x = 0$, or than the root of the cubick equation $x^3 - 80x^2 - 1998x = 14{,}937$. Q. E. D.

And it is evident that the ſame things will take place in any other equation whatſoever, in which the term that involves the higheſt power of x is greater than the ſum of all the other terms of the equation that involve the other, or inferiour, powers of x, and it's exceſs above the ſaid ſum is equal to the abſolute term, or in which all the terms on the left-hand ſide of the equation, except that which involves the higheſt power of x, are ſubtracted from that term, and conſequently marked with the ſign —.

Art. 6. In the third place, when the term that involves the higheſt power of x in the equation is added to the next term, or term involving the next power of x, or to the two next, or the three next terms, or any greater number of the terms next following it, and their ſum is greater than the ſum of all the remaining terms on the left-hand ſide of the equation, or than all the other terms that involve the lower powers of x, and the latter

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ſum is ſubtracted from the former, the equation can have but one root.

Thus, for example, the equation $x^4 + 80x^3 - 1998x^2 - 14{,}937x = 5000$, and the equation $x^4 + 80x^3 + 1998x^2 - 14{,}937x = 5000$, can have but one root a-piece. And all the following equations will have but one root a piece, to wit,

$$x^7 + 3x^6 - 5x^5 - 7x^4 - 9x^3 - 11x^2 - 13x = 15,$$
and $x^7 + 3x^6 + 5x^5 - 7x^4 - 9x^3 - 11x^2 - 13x = 15,$
and $x^7 + 3x^6 + 5x^5 + 7x^4 - 9x^3 - 11x^2 - 13x = 15,$
and $x^7 + 3x^6 + 5x^5 + 7x^4 + 9x^3 - 11x^2 - 13x = 15,$
and $x^6 + 3x^5 + 5x^4 + 7x^3 + 9x^2 + 11x - 13x = 15.$

This will eaſily appear from reaſonings concerning the manner in which the ſeveral terms involving the powers of x increaſe, and in which the exceſs of the ſum of the terms involving the higher powers of x above the ſum of the terms involving the lower powers of x increaſes, exactly ſimilar to the reaſonings uſed in the foregoing article. And therefore I do not think it neceſſary to be more particular in the proof of it.

Art. 7. And there are ſome other caſes in which we may be ſure before-hand that an equation has only one real and affirmative root, though the ſigns of the terms on the left-hand ſide of the equation ſhould vary from + to — and from — to + two or three times, or oftener. But theſe caſes are more difficult to diſtinguiſh and aſcertain than the three caſes before-mentioned. And therefore I ſhall ſay nothing further concerning them.

Art. 8. In all the caſes above-deſcribed, and in all others where we know before-hand that the equation that is to be reſolved has only one real and affirmative root, the beſt way of beginning the inveſtigation of the value of it's root, in order to approximate further to it afterwards by Mr. Raphſon's method above-mentioned, (if the conditions of the problem from which the equation is derived do not themſelves point out certain limits of the magnitude of the ſaid root, which will afford us a ſufficiently near value of it to begin our approximation with, which they often will be found to do,) will be that invented by Stevinus, a mathematician of Bruges in the Spaniſh Netherlands, who flouriſhed in the beginning of the laſt century, and died in the year 1633. This method is as follows:

Since the propoſed equation can have but one real and affirmative root, it is evident that the whole compound quantity which forms the left-hand ſide of the equation will increaſe continually, or without ever decreaſing, at the ſame time that x increaſes. Therefore, if we ſubſtitute any particular value for x in the ſaid compound quantity, and we find that the value of the ſaid compound quantity ariſing from ſuch ſubſtitution is greater than the abſolute term of the propoſed equation, we may conclude with certainty that the ſaid ſubſtituted value of x is greater than the value of x in the propoſed equation, or than the root of the ſaid equation: and, if the value of the ſaid compound quantity ariſing from ſuch ſubſtitution is leſs than the abſolute term of the equation, we

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may conclude with certainty that the ſaid ſubſtituted value of x is leſs than the root of the ſaid equation. Let us therefore firſt ſubſtitute 1 inſtead of x in the ſaid compound quantity, and try what the reſult will be; and, if the reſult is much leſs than the abſolute term of the equation, and conſequently 1 is much leſs than the true root of the equation, let us in the next place ſuppoſe x to be equal to 10, or to 9, or to 8, or to 7, or ſuch other number conſiſting of one or at moſt two figures, as we conjecture to be neareſt to the true value of x in the propoſed equation; and let this ſecond conjectural value of x be ſubſtituted inſtead of x in the ſaid compound quantity which forms the left-hand ſide of the equation. And thus, by a few eaſy trials, we ſhall ſoon find a value of x that will be true to at leaſt one figure, and without much difficulty we may find a value of it that ſhall be true in the two firſt places of figures; and we ſhall know at the ſame time whether the ſaid value is greater or leſs than the true value of x in the propoſed equation; becauſe it will be greater, if the reſult of the ſubſtitution of it in the aforeſaid compound quantity, which forms the left-hand ſide of the equation, is greater than the abſolute term of the equation; and it will be leſs, if the ſaid reſult is leſs than the ſaid abſolute term. And from this firſt near value of x, ſo obtained by theſe eaſy trials, or ſubſtitutions, we may begin our approximation to a more exact value of x in the manner deſcribed by Mr. Raphſon, and which has been exemplified above in the Appendix to Dr. Halley's Tract. Nothing more therefore need be ſaid on the re-

ſolution of equations of this kind, which we know beforehand to have but one real and affirmative root.

The foregoing method of finding the two firſt figures of the root of an equation of this kind is very fully illuſtrated by judicious examples by the learned Mr. John Kerſey, in his excellent Treatiſe of Algebra, publiſhed in the latter part of the laſt century, in the years 1673 and 1674, in two volumes, folio, than which, I believe, there is not a better Treatiſe on that ſubject to be met with *. See Vol. I, Book II, Chapter 10, pages 265, 266, 267; which 10th chapter is alſo reprinted above in the foregoing part of this preſent work, pages 325, 326, &c - - - 338.

Art. 9. But, when we meet with a cubick, or a biquadratick, equation, or an equation of any higher order, in which there are two, or more, changes of the ſigns + and − that are prefixed to the terms that involve the powers of x, and in which conſequently there are, or, at leaſt, may be, for aught we know to the contrary, two or more real and affirmative roots, or different values of x,

* The learned Dr. John Wallis, in the Preface to the Latin Edition of his Algebra, in the 2nd volume of the Collection of all his Works publiſhed at Oxford in the year 1693, ſpeaks of Mr. Kerſey's Algebra in theſe words: *Suaſerim ut Lector conſulat ex noſtris (præter alios,) Kerſæum noſtrum; qui duobus voluminibus integram Algebræ tractationem exhibuit, fusè quidem at perſpicuè traditam. Quo nemo feliciùs quæſtiones Diophanteas elucidavit.*

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which, being ſubſtituted inſtead of x in the terms that form the left-hand ſide of the equation, will make the reſult of them equal to the right-hand ſide of it, or it's abſolute, or known, term, it is often difficult to know which root we had beſt begin to inveſtigate. And, if we make conjectures concerning the middle values of x, or if we make conjectures concerning it's values in general;—and the quantities we pitch upon for one of it's values are greater than it's leaſt value and nearer to one of it's middle values;—and we ſubſtitute one of theſe quantities inſtead of x in the compound quantity that forms the left-hand ſide of the equation;—and we find, after ſuch ſubſtitution, that the value of the ſaid compound quantity thence ariſing is greater than the abſolute term of the equation;—we ſhall not be able to conclude from thence that the ſaid conjectural value of x is greater than it's true value; nor, *vice verſâ*, if the value of the ſaid compound quantity reſulting from ſuch ſubſtitution is leſs than the abſolute term of the equation, ſhall we be able to conclude that the ſaid conjectural value of x is leſs than it's true value. For, in theſe equations which have more than one real and affirmative root, the compound quantity which forms the left-hand ſide of the equation will ſometimes decreaſe at the ſame time that x increaſes, as may be ſeen in Chap. X, pages 71, 72, &c - - - 90, of my Diſſertation on the Uſe of the Negative Sign in Algebra. In theſe equations therefore, which have, or ſeem to have, more than one real and affirmative root, there is often a good deal of difficulty in knowing how to ſet about the reſolution of them by Mr. Raphſon's method of approximation. And, I believe, it

will uſually be found expedient in theſe caſes to confine ourſelves to the inveſtigation of the leaſt root, or value of x, in the equation. For, from the foregoing reaſonings concerning the increaſe of x from 0 *ad infinitum*, and the contemporary increaſe of each of the terms in the equation that involves x, and the contemporary increaſe, or decreaſe, of the whole compound quantity that forms the left-hand ſide of the equation, it is evident that in the firſt part of the increaſe of x from 0 *ad infinitum*, or while it increaſes from 0 till it becomes equal to the leaſt root of the propoſed equation, the ſaid compound quantity will increaſe with it continually, or without ever decreaſing, though afterwards, when x is greater than the ſaid leaſt root, the ſaid compound quantity will, if the equation has more than one real and affirmative root, ſometimes decreaſe while x increaſes. This circumſtance will enable us to make our conjectural approaches to the value of the ſaid leaſt root of the equation with more facility and perſpicuity than to the other roots of it, and will enable us to conclude with certainty whether the value we have either aſſumed by a conjecture, or found by any previous method of inveſtigation, for the said leaſt root of the equation, or leaſt value of x, is greater, or leſs, than it's true value. And on this account I am much inclined to think that the beſt way of reſolving an equation of this kind is to begin with the inveſtigation of it's leaſt root. And the method I would recommend for this purpoſe is a very ſimple and eaſy one, being grounded on the following obvious propoſition:

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Art. 10. If there be any equation involving ſeveral different powers of the unknown quantity x that are connected with each other both by addition and ſubtraction, or by both the ſigns + and −, ſuch as the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$; and there be another equation that has the ſame abſolute term r as the former equation, but has only ſo many terms in the left-hand ſide of it as there are terms in the left-hand ſide of the former equation connected together by the ſign +, which are four, to wit, x^7, lx^5, nx^3, and qx; and, if the unknown quantity contained in this ſecond equation be called y, and the powers of y involved in theſe terms of the ſecond equation are the ſame as the powers of x involved in the ſaid terms of the firſt equation which are connected together by the ſign +; and, if the co-efficients of the ſaid powers of y in the ſecond equation are the ſame as the co-efficients of the ſame powers of x in the firſt equation; and, laſtly, if all theſe terms of the ſaid ſecond equation involving the ſaid powers of y be added to each other, or connected together by the ſign +: So that the ſaid ſecond equation ſhall be $y^7 + ly^5 + ny^3 + qy = r$:—Then will the value of y, or the only root of the ſecond equation $y^7 + ly^5 + ny^3 + qy = r$, be leſs than the leaſt root of the firſt equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

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DEMONSTRATION.

If there be two equations $x^7 + lx^5 + nx^3 + qx = r + D$, and $y^7 + ly^5 + ny^3 + qy = r$, in both which the terms which form the left-hand ſides of the equations are connected together by the ſign +, and involve the very ſame powers of the two unknown quantities x and y multiplied into the very ſame co-efficients 1, l, n, and q, but the abſolute term of the former equation exceeds the abſolute term of the latter equation by any given quantity called D, it is evident, from art. 3, that x, or the only root of the former equation $x^7 + lx^5 + nx^3 + qx = r + D$, will be greater than y, or the only root of the latter equation $y^7 + ly^5 + ny^3 + qy = r$.

Now let the three terms kx^6, mx^4, and px^2 in the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, which are marked with the ſign —, be added to both ſides of the equation. Then, it is evident, we ſhall have $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$; which will have the very ſame roots as the former equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, becauſe it is only the ſame equation under another form. Therefore the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ will be alſo the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Let the leaſt root of this equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$ be called a, and it's other

roots,

roots, if it has any other roots, be denoted by the letters b, c, d, &c.

Then it is evident that the trinomial quantity $kx^6 + mx^4 + px^2$ will have as many different values as there are different values of x, or different roots of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, or of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, and will be equal either to the trinomial quantity $ka^6 + ma^4 + pa^2$, or to the trinomial quantity $kb^6 + mb^4 + pb^2$, or to the trinomial quantity $kc^6 + mc^4 + pc^2$, or to the trinomial quantity $kd^6 + md^4 + pd^2$, &c; and the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ will repreſent as many different equations, having one root a-piece, as it has different roots, to wit, 1ſt, the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, which will have only one root, to wit, a; and, 2dly, the equation $x^7 + lx^5 + nx^3 + qx = r + kb^6 + mb^4 + pb^2$, which will have only one root, to wit, b; and, in like manner, the equations $x^7 + lx^5 + nx^3 + qx = r + kc^6 + mc^4 + pc^2$, and $x^7 + lx^5 + nx^3 + qx = r + kd^6 + md^4 + pd^2$, &c, which have each only one root, to wit, c, and d, &c, reſpectively. Therefore, if we ſubſtitute, inſtead of the trinomial quantity $kx^6 + mx^4 + px^2$, any one of it's particular values, $ka^6 + ma^4 + pa^2$, or $kb^6 + mb^4 + pb^2$, or $kc^6 + mc^4 + pc^2$, or $kd^6 + md^4 + pd^2$, &c, in the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px$, the equation thence ariſing will not have ſeveral different roots, as the ſaid equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$ had, but only one root, to wit, that root of the ſaid general equation which was ſubſtituted inſtead of x in the trinomial quantity $kx^6 + mx^4 + px^2$.

Let us therefore ſuppoſe the trinomial quantity $kx^6 + mx^4 + px^2$ to be equal to the trinomial quantity $ka^6 + ma^4 + pa^2$, which is it's leaſt value; and let $ka^6 + ma^4 + pa^2$ be denoted by the letter D.

Then will $x^7 + lx^5 + nx^3 + qx$ be $= r + D$; and this equation will have but one root, which will be equal to a, or the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

But it has been obſerved above, that the root of the equation $x^7 + lx^5 + nx^3 + qx = r + D$ will be greater than the root of the equation $y^7 + ly^5 + ny^3 + qy = r$.

Therefore the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, will be greater than the root of the equation $y^7 + ly^5 + ny^3 + qx = r$; or the root of the equation $y^7 + ly^5 + ny^3 + qy = r$ will be leſs than the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Q. E. D.

The Application of the foregoing Lemma to the Inveſtigation of a near Value of the Leaſt Root of an Equation that has more than One real and affirmative Root.

Art. 11. This Lemma being perfectly underſtood, I would propoſe to begin the reſolution of any equation of

the foregoing form $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, (or in which there are two, or more changes of the ſigns, + and —, of the terms that involve the unknown quantity x,) by finding the ſole root of the equation $y^7 + ly^5 + ny^3 + qy = r$, which is derived from the former equation by dropping all the terms that have the ſign — prefixed to them, and by ſubſtituting y inſtead of x in the remaining terms. This root I would find in the manner preſcribed by Stevinus and Kerſey, to about two places of figures, and would denote it by the ſmall Greek letter α, to diſtinguiſh it from a, or the true value of the leaſt root of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, or $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the ſole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$; than which, by the foregoing Lemma, it would neceſſarily be leſs. And this quantity α I would conſider as the firſt near value of a, or the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, which we are ſeeking.

Art. 12. In the next place I would ſubſtitute α inſtead of x in the compound quantity $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx$, (which forms the left-hand ſide of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$,) in order to find how far the value of the ſaid compound quantity reſulting from this ſubſtitution, or how far the compound quantity $\alpha^7 - k\alpha^6 + l\alpha^5 - m\alpha^4 + n\alpha^3 - p\alpha^2 + q\alpha$, (which will neceſſarily be leſs than the abſolute term r,) will fall ſhort of the abſolute term r. And, if I found that the ſaid compound quantity

tity $a^7 - ka^6 + la^5 - ma^4 + na^3 - pa^2 + qa$ was conſiderably leſs than the abſolute term r, as, for example, not more than two third parts of it, I would increaſe the abſolute term r of the equation $y^7 + ly^5 + ny^3 + qy = r$ by the addition of the three terms ka^6, ma^4, and pa^2, and would ſeek the root of the new equation thence ariſing, to wit, the equation $y^7 + ly^5 + ny^3 + qy = r + ka^6 + ma^4 + pa$, to two places of figures, by the method of Stevinus and Kerſey, in the ſame manner as we had before found the root of the equation $y^7 + ly^5 + ny^3 + qy = r$; and this root I would denote by the ſmall Greek letter ϵ.

Art. 13. This quantity ϵ, being the ſole root of the equation $y^7 + ly^5 + ny^3 + qy = r + ka^6 + ma^4 + pa^2$, would be greater than a, or the ſole root of the equation $y^7 + ly^5 + ny^3 + qy = r$, becauſe the abſolute term $r + ka^6 + ma^4 + pa^2$ of the former equation is greater than r, or the abſolute term of the latter equation. And it will be leſs than α, or the ſole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, becauſe the abſolute term $r + ka^6 + ma^4 + pa^2$ is leſs than the abſolute term $r + k\alpha^6 + m\alpha^4 + p\alpha^2$. Therefore it will be of an intermediate magnitude between α and a, and will approach nearer to a than the quantity α did, and conſequently may juſtly be called the ſecond near value of a, or of the ſole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, or of the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$.

Art. 14.

Art. 14. Having thus found theſe two near values, a and c, of a, or the leaſt root of the original equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, by reſolving the two equations $y^7 + ly^5 + ny^3 + qy = r$ and $y^7 + ly^5 + ny^3 + qy = r + ka^6 + ma^4 + pa^2$ in the manner preſcribed by Stevinus and Kerſey, I would denote the difference by which the ſecond of theſe near values of a, to wit, c, falls ſhort of a, or the leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, by the letter z, and would ſubſtitute $c + z$ inſtead of x in the ſaid equation, omitting all the terms that would involve in them any powers of z except the ſimple power of it, or z itſelf, agreeably to the directions of Mr. Raphſon in his *Analyſis Æquationum Univerſalis* above-mentioned, and would reſolve the new equation reſulting from ſuch ſubſtitution, as a ſimple equation; by which we ſhould obtain a near value of z, which, being added to c, would give us the value of $c + z$, or a third near value of a, or of the ſole root of the equation $x^7 + lx^5 + nx^3 + qx = r + ka^6 + ma^4 + pa^2$, or of the leaſt root of the equation $x^7 + lx^5 + nx^3 + qx = r + kx^6 + mx^4 + px^2$, or of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$. And then we might go on to find a fourth, and a fifth, and other following near values of a, or the ſaid leaſt root of the equation $x^7 - kx^6 + lx^5 - mx^4 + nx^3 - px^2 + qx = r$, by Mr. Raphſon's method of approximation, in the manner exemplified above in the Appendix to Dr. Halley's Tract, till we had found it to as great degree of exactneſs as we deſired.

This method of proceeding will be better underſtood by

by applying it to the reſolution of two or three particular equations.

An Example of the Reſolution of a Cubick Equation by the Method above-deſcribed.

Art. 15. Let it be required to find the leaſt root of the cubick equation $x^3 - 39x^2 + 479x = 1881$.

This equation has three real and affirmative roots, of which we will now endeavour to find the leaſt in the manner juſt now deſcribed.

In the firſt place, we muſt ſeek by the method of Stevinus and Kerſey, the ſole root of the equation $y^3 + 479y = 1881$; which will be leſs than the leaſt root of the propoſed equation $x^3 - 39x^2 + 479x = 1881$, or $x^3 + 479x = 1881 + 39x^2$, or (if we put a for the leaſt value of x in this equation,) than the ſole root of the equation $x^3 + 479x = 1881 + 39a^2$. Now the ſole root of the equation $y^3 + 479y = 1881$ may be found by the method preſcribed by Stevinus and Kerſey, by proceeding as follows:

If y is $= 1$, the compound quantity $y^3 + 479y$ will be $(= 1 + 479) = 480$; which is leſs than 1881. Therefore 1 will be leſs than the true value of y.

Secondly,

Secondly, if y is $= 10$, the compound quantity $y^3 + 479y$ will be $(= 1000 + 4790) = 5790$; which is greater than 1881. Therefore 10 will be greater than the true value of y.

Thirdly, ſince y is greater than $1\frac{1}{2}$ but leſs than 10, let us ſuppoſe it to be equal to 5. Then we ſhall have $y^3 + 479y$ $(= 125 + 479 \times 5 = 125 + 2395)$ $= 2520$; which is greater than 1881. Therefore 5 will be greater than the true value of y.

Fourthly, ſince y is leſs than 5, let us ſuppoſe it to be $= 4$. Then we ſhall have $y^3 + 479y$ $(= 64 + 479 \times 4 = 64 + 1916) = 1980$; which is greater than 1881. Therefore 4 is greater than the true value of y. But the difference cannot be great.

Fifthly, ſince y is leſs than 4, but pretty nearly equal to it, let us ſuppoſe it to be $= 3.8$. Then we ſhall have $y^3 + 479y$ $(= \overline{3.8}|^3 + 479 \times 3.8 = 54.872 + 1820.2)$ $= 1875.072$; which is a little leſs than 1881. Therefore 3.8 will be a little leſs than the true value of y in the equation $y^3 + 479y = 1881$. This value of y, or of the ſole root of the cubick equation $y^3 + 479y = 1881$, is near enough to it's true value for our preſent purpoſe. We muſt therefore conſider 3.8 as the firſt near value of a, or of the leaſt root of the original equation $x^3 - 39xx + 479x = 1881$, or of the equation $x^3 + 479x = 1881 + 39xx$, or of the ſole root of the equation $x^3 + 479x = 1881 + 39a^2$. This firſt near value of the ſaid root, or a, we will denote by the ſmall Greek letter α.

Art. 16. We muſt now proceed to find the value of the ſole root of the equation $y^3 + 479y = 1881 + 39a^2$, (or $y^3 + 479y = 1881 + 39 \times 3.61^2$, or $y^3 + 479y = 1881 + 39 \times 14.44$, or $y^3 + 479y = 1881 + 563.16$,) or $y^3 + 479y = 2444.16$ by the method of Stevinus and Kerſey.

Now we have ſeen already that, if y is $= 5$, we ſhall have $y^3 + 479y = 2520$; which is greater than 2444.16. Therefore 5 will be greater than the true value of y. But the difference between them will be but ſmall; ſo that there will be no occaſion to determine the value of y in this equation to any greater degree of exactneſs. We may therefore conſider 5 as the ſecond near value of x, or of the leaſt root of the original equation $x^3 - 39x^2 + 479x = 1881$, or of the equation $x^3 + 479x = 1881 + 39x^2$, or of the ſole root of the equation $x^3 + 479x = 1881 + 39a^2$. This ſecond near value of the ſaid root, or a, we will denote by the ſmall Greek letter ϵ.

Art. 17. Having thus obtained 5 for the ſecond near value of the leaſt root of the original equation $x^3 - 39xx + 479x = 1881$, we may proceed to inveſtigate the ſaid root to a greater degree of exactneſs by Mr. Raphſon's method of approximation, by putting z for the difference by which 5 falls ſhort of the true value of the ſaid leaſt root, and ſubſtituting $5 + z$ inſtead of x in the ſaid equation, with an omiſſion of all the terms which involve any powers of z, except the ſimple power of it, or z itſelf. This may be done as follows:

Since x is $= 5 + z$, we shall have

$$xx\ (= \overline{5 + z}^2 = \overline{5}^2 + 2 \times 5z + \&c) = 25 + 10z + \&c,$$

and $x^3\ (= \overline{5 + z}^3 = \overline{5}^3 + 3 \times \overline{5}^2 \times z + \&c = 125 + 3 \times 25 \times z + \&c) = 125 + 75z + \&c,$

and $39xx\ (= 39 \times \overline{25 + 10z + \&c} = 39 \times 25 + 39 \times 10z + \&c) = 975 + 390 \times z + \&c,$

and $479x\ (= 479 \times \overline{5 + z} = 479 \times 5 + 479 \times z) = 2395 + 479z,$

and consequently $x^3 - 39x^2 + 479x =$

$$\left\{\begin{array}{rrr} & 125 + & 75z + \&c \\ - & 975 - & 390z - \&c \\ + & 2395 + & 479z \end{array}\right\}$$

$$= \left\{\begin{array}{rrr} & 2520 + & 554z + \&c \\ - & 975 - & 390z - \&c \end{array}\right\}$$

$$= 1545 + 164z \ \&c.$$

But $x^3 - 39x^2 + 479x$ is $= 1881.$

Therefore $1545 + 164z$ &c will also be $= 1881.$ And consequently $164z$ will be $(= 1881 - 1545) = 336,$ and z will be $= \frac{336}{164} = 2.0$ &c.

Therefore x, or $5 + z$, will be $= 5 + 2.0$ &c $= 7.0$ &c, or will be greater than 7.

Art. 18. Now let 7 be ſubſtituted inſtead of x in the compound quantity $x^3 - 39x^2 + 479x$, in order to diſcover whether the reſult will be greater, or leſs, than 1881, or the abſolute term of the equation $x^3 - 39x^2 + 47x = 1881$; and conſequently whether 7 is greater, or leſs, than the true value of the leaſt root of the ſaid equation. By this ſubſtitution we ſhall have $x^3 - 39x^2 + 479x$ $(= \overline{7}|^3 - 39 \times \overline{7}|^2 + 479 \times 7 = 343 - 39 \times 49 + 3353 = 3696 - 1911) = 1785$; which is leſs than 1881. Therefore 7 is leſs than the true value of the leaſt root of the propoſed equation $x^3 - 39x^2 + 479x = 1881$.

Art. 19. Let us therefore ſuppoſe x to be $= 7 + v$, and ſubſtitute $7 + v$ inſtead of x in the equation $x^3 - 39x^2 + 479x = 1881$. And we ſhall have

$$xx\ (= \overline{7 + v}^2) = \overline{7}|^2 + 2 \times 7 \times v + \&c = 49 + 14v + \&c,$$

$$\text{and } x^3\ (= \overline{7 + v}^3 = \overline{7}|^3 + 3 \times \overline{7}|^2 \times v + \&c = 343 + 3 \times 49 \times v + \&c) = 343 + 147 \times v + \&c,$$

$$\text{and } 39xx\ (= 39 \times \overline{49 + 14v + \&c} = 39 \times 49 + 39 \times 14v + \&c) = 1911 + 546 \times v + \&c,$$

$$\text{and } 479x\ (= 479 \times \overline{7 + v} = 479 \times 7 + 479 \times v) = 3353 + 479v,$$

and conſequently $x^3 - 39xx + 479x =$

$$\left\{\begin{array}{r} 343 + 147 \times v + \&c \\ -\ 1911 - 546 \times v - \&c \\ +\ 3353 + 479 \times v + \&c \end{array}\right\}$$

$$= \left\{\begin{array}{r} 3696 + 626 \times v + \&c \\ -\ 1911 - 546 \times v - \&c \end{array}\right\}$$

$$= 1785 + 80 \times v \ \&c.$$

But $x^3 - 39xx + 479x$ is $= 1881$.

Therefore $1785 + 80 \times v$ &c will alſo be $= 1881$. And conſequently $80 \times v$ will be $(= 1881 - 1785) = 96$, and v will be $= \frac{96}{80} = 1.2$. Therefore x, or $7 + v$, will be $(= 7 + 1.2) = 8.2$.

Art. 20. Now let 8.2 be ſubſtituted inſtead of x in the compound quantity $x^3 - 39x^2 + 479x$, in order to diſcover whether the reſult will be greater, or leſs, than 1881, or the abſolute term of the propoſed equation $x^3 - 39xx + 479x = 1881$, and conſequently whether 8.2 is greater, or leſs, than the true value of the leaſt root of that equation.

By this ſubſtitution we ſhall have $x^3 - 39xx + 479x$ $(= \overline{8.2}^3 - 39 \times \overline{8.2}^2 + 479 \times 8.2 = 551.368 - 39 \times 67.24 + 479 \times 8.2 = 551.368 - 2622.36 + 3927.8 = 4479.168 - 2622.36) = 1856.808$; which is leſs than 1881. Therefore 8.2 is leſs than the true value of the leaſt root of the equation $x^3 - 39xx + 479x = 1881$.

$x = 8.8 + \varphi$,

and $xx\ (= \overline{8.8 + \varphi}^2 = \overline{8.8}^2 + 2 \times 8.8 \times \varphi + \&c)$
$= 77.44 + 17.6 \times \varphi + \&c$,

and $x^3\ (= \overline{8.8 + \varphi}^3 = \overline{8.8}^3 \times 3 \times \overline{8.8}^2 \times \varphi + \&c$
$= 681.472 + 3 \times 77.44 \times \varphi + \&c)$
$= 681.472 + 232.32 \times \varphi + \&c$,

and $39xx\ (= 39 \times \overline{77.44 + 17.6 \times \varphi + \&c} = 39 \times$
$77.44 + 39 \times 17.6 \times \varphi + \&c) =$
$3020.16 + 686.4 \times \varphi + \&c$,

and $479x\ (= 479 \times \overline{8.8 + \varphi} = 479 \times 8.8 + 479 \times \varphi)$
$= 4215.2 + 479 \times \varphi$,

and consequently $x^3 - 39xx + 479x =$

$$\left\{\begin{array}{rrrrr} 681.472 & + & 232.32 & \times \varphi & + \&c \\ - 3020.16 & - & 686.4 & \times \varphi & - \&c \\ + 4215.2 & + & 479 & \times \varphi & + \&c \end{array}\right\} =$$

$$\left\{\begin{array}{rrrrr} 4896.672 & + & 711.32 & \times \varphi & + \&c \\ - 3020.16 & - & 686.4 & \times \varphi & - \&c \end{array}\right\}$$

$= 1876.512 + 24.92 \times \varphi \quad \&c.$

But $x^3 - 39xx + 479x$ is $= 1881$.

Therefore $1876.512 + 24.92 \times \varphi$ will alſo be $= 1881$.

Therefore $24.92 \times \varphi$ will be $(= 1881 - 1876.512) = 4.488$; and conſequently φ will be $= \frac{4.488}{24.92} = 0.18$.

Therefore

Therefore x, or $8.8 + \varphi$, will be $(= 8.8 + 0.18) = 8.98$, or the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$ will be $= 8.98$. Q. E. I.

Art. 24. This value of the ſaid leaſt root will be very little leſs than the truth. For, if we ſuppoſe x to be $= 8.98$, we ſhall have xx $(= \overline{8.98}^2) = 80.6404$, and x^3 $(= \overline{8.98}^3) = 724.150,792$, and $39xx$ $(= 39 \times 80.6404) = 3144.9796$, and $479x$ $(= 479 \times 8.98) = 4301.42$, and conſequently $x^3 - 39xx + 479x$ $(= 724.150,792 - 3144.9796 + 4301.42 = 5025.570,792 - 3144.9796) = 1880.594,192$; which is very little leſs than 1881, or the abſolute term of the propoſed equation $x^3 - 39xx + 479x = 1881$. And conſequently 8.98 muſt be very little leſs than the true value of the leaſt root of the ſaid equation.

Art. 25. Since 8.98 is very little leſs than the true value of a, or the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$, it ſeems reaſonable to conjecture that it's true value may be the whole number 9. And ſo, upon trial, we ſhall find it to be. For, if we ſuppoſe x to be $= 9$, we ſhall have $xx = 81$, and $x^3 = 729$, and $39xx$ $(= 39 \times 81) = 3159$, and $479x$ $(= 479 \times 9) = 4311$, and conſequently $x^3 - 39xx + 479x$ $(= 729 - 3159 + 4311 = 5040 - 3159) = 1881$. Therefore 9 is the true value of a, or of the leaſt root of the propoſed equation $x^3 - 39xx + 479x = 1881$. Q. E. D.

Art. 26.

Art. 26. Having thus found one of the roots of the cubick equation $x^3 - 39nx + 479x = 1881$, we may reduce the equation to a lower degree, or convert it into a quadratick equation of which the roots ſhall be the ſame with the remaining roots of the ſaid cubick equation, by proceeding in the manner following:

Let a be put, as before, for 9, or the leaſt root of the ſaid equation; and let x repreſent each of the other two roots of it, if it has two other roots, or the only remaining root of it, if it has but one other root.

Then, ſince $a^3 - 39aa + 479a$ is $= 1881$, and $x^3 - 39xx + 479x$ is alſo $= 1881$, we ſhall have $a^3 - 39aa + 479a = x^3 - 39xx + 479x$, and conſequently (adding $39xx$ to both ſides,) $a^3 + 479a + 39xx - 39aa = x^3 + 479x$, and (ſubtracting $a^3 + 479a$ from both ſides of the equation,) $39xx - 39aa = x^3 - a^3 + 479x - 479a$, or $39 \times \overline{xx - aa} = x^3 - a^3 + 479 \times \overline{x - a}$, and (dividing both ſides of the equation by the reſidual quantity $x - a$,) $39 \times \overline{x + a} = xx + xa + aa + 479$, or $39x + 39a = xx + ax + aa + 479$, and (ſubtracting $xx + ax$ from both ſides,) $39x + 39a - ax - xx = aa + 479$, and (ſubtracting $39a$ from both ſides,) $39x - ax - xx = 479 + aa - 39a$, or (becauſe a is $= 9$,) $39x - 9x - xx = 479 + 81 - 39 \times 9$, or $30x - xx$ $(= 560 - 39 \times 9 = 560 - 351)$ $= 209$; which is only a quadratick equation.

Q. E. F.

Art. 27

Art. 27. This quadratick equation, $30x - xx = 209$, will have two roots, which may be found in the manner following:

Subtract both ſides of the equation $30x - xx = 209$ from 225, or the ſquare of 15, or of $\frac{30}{2}$, or of half the co-efficient of x in the ſaid equation. And we ſhall have $225 - 30x + xx\ (= 225 - 209) = 16$. Therefore the ſquare-root of the trinomial quantity $225 - 30x + xx$ will be equal to the ſquare-root of 16, that is, to 4. But the trinomial quantity $225 - 30x + xx$ has two ſquare-roots, to wit, $15 - x$ and $x - 15$. Therefore $15 - x$ will be $= 4$, and conſequently 15 will be $= 4 + x$, and x will be $= 15 - 4 = 11$. And $x - 15$ will alſo be $= 4$, and conſequently x will be $= 4 + 15 = 19$. Therefore 11 and 19 will be the two roots of the quadratick equation $30x - xx = 209$, and conſequently will be the middle and greateſt roots of the propoſed cubick equation $x^3 - 39xx + 479x = 1881$.

Q. E. I.

And ſo, upon trial, we ſhall find them to be. For, if we ſuppoſe x to be $= 11$, we ſhall have $xx = 121$, and $x^3 = 1331$, and $39xx\ (= 39 \times 121) = 4719$, and $479x\ (= 479 \times 11) = 5269$, and conſequently $x^3 - 39xx + 479x\ (= 1331 - 4719 + 5269 = 6600 - 4719) = 1881$.

And, if we ſuppoſe x to be $= 19$, we ſhall have $xx = 361$, and $x^3 = 6859$, and $39xx\ (= 39 \times 361)$ =

$= 14{,}079$, and $479x$ $(= 479 \times 19) = 9101$, and conſequently $x^3 - 39xx + 479x$ $(= 6859 - 14{,}079 + 9101 = 15{,}960 - 14{,}079) = 1881$.

Therefore both 11 and 19 are roots of the propoſed equation $x^3 - 39xx + 479x = 1881$, and conſequently the three roots of that equation are 9, 11, and 19.

Q. E. I.

Another Example of the Reſolution of an Equation that, by the Form of it, or the Changes of the Signs + and — prefixed to it's Terms, ſeems capable of having more than One real and affirmative Root, by means of the foregoing Proceſſes deſcribed above in Art. 9, 10, 11, 12, *and* 13.

Art. 28. Let it be required to reſolve the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$ (which riſes to the fifth power of x,) by means of the ſaid proceſſes.

This equation is of ſuch a form as to be capable of having three real and affirmative roots, if the co-efficients of x^5, x^4, x^3, and x, to wit, 1, 7, 20, and 155, and the abſolute term 10,000, are of the proper relative magnitudes with reſpect to each other for that purpoſe. But of theſe roots, (whether they be three in number, or fewer than three,) I now propoſe to inveſtigate only one root;

root; and that will be the leaſt of them, if there are more than one: and this root I ſhall denote by the letter a.

Art. 29. Now, ſince $x^5 - 7x^4 + 20x^3 - 155xx$ is $= 10{,}000$, we ſhall have $x^5 + 20x^3 = 10{,}000 + 7x^4 + 155xx$; and conſequently (becauſe a is equal to one of the values of x, to wit, the leaſt value of it, if it has more than one value,) $a^5 + 20a^3$ will be $= 10{,}000 + 7a^4 + 155aa$. We muſt therefore endeavour to find the root a of this equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$.

Now the root of this equation will be greater than the root of the equation $y^5 + 20y^3 = 10{,}000$, becauſe the trinomial quantity $10{,}000 + 7a^4 + 155aa$ is greater than the ſingle quantity 10,000. We will therefore, as a firſt approximation to the value of a, or the root of the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$, ſeek the value of y, or the root of the equation $y^5 + 20y^3 = 10{,}000$.

Art. 30. Now, if we ſuppoſe y to be equal to 1, we ſhall have $y^3 = 1$, and $y^5 = 1$, and $20y^3$ $(= 20 \times 1)$ $= 20$, and conſequently $y^5 + 20y^3$ $(= 1 + 20) = 21$; which is much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

Secondly, if we ſuppoſe y to be $= 10$, we ſhall have $y^3 = 1000$, and $y^5 = 100{,}000$, and $20y^3 = 20{,}000$, and

$y^5 +$

$y^5 + 20y^3$ (= 100,000 + 20,000) = 120,000; which is much greater than the abſolute term 10,000. Therefore 10 muſt be much greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the 3d place, ſuppoſe y to be equal to 5. And then we ſhall have $y^3 = 125$, and $y^5 = 3125$, and $20y^3$ (= 20 × 125) = 2500, and $y^5 + 20y^3$ (= 3125 + 2500) = 5625; which is leſs than the abſolute term 10,000. Therefore 5 muſt be leſs than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the 4th place, ſuppoſe y to be equal to 6. And then we ſhall have $y^3 = 216$, and $y^5 = 7776$, and $20y^3$ (= 20 × 216) = 4320, and conſequently $y^5 + 20y^3$ (= 7776 + 4320) = 12,096; which is greater than the abſolute term 10,000. Therefore 6 muſt be greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the fifth place, ſuppoſe y to be equal to 5.8. And then we ſhall have $y^3 = 195.112$, and $y^5 = 6563.56768$, and $20y^3$ (= 20 × 195.112) = 3902.240, and conſequently $y^5 + 20y^3$ (= 6563.56768 + 3902.240) = 10,465.80768; which is ſomewhat greater than the abſolute term 10,000. Therefore 5.8 muſt be ſomewhat greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$. But the difference will be but ſmall, and conſequently we may conſider 5.8 as the root of the ſaid equation $y^5 + 20y^3 = 10{,}000$, and

as the firſt near value of a in the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$.

Art. 31. Let this quantity 5.8 be called α. And let us, for a ſecond near value of a, find the root of the equation $y^5 + 20y^3 = 10{,}000 + 7\alpha^4 + 155\alpha^2$. For this root will be greater than α, or 5.8, or the root of the equation $y^5 + 20y^3 = 10{,}000$, becauſe the trinomial quantity $10{,}000 + 7\alpha^4 + 155\alpha^2$ is greater than the ſingle quantity 10,000; but it will be leſs than a, or the root of the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$, becauſe the trinomial quantity $10{,}000 + 7\alpha^4 + 155\alpha^2$ is leſs than the trinomial quantity $10{,}000 + 7a^4 + 155aa$.

Now, ſince α is $= 5.8$, we ſhall have $\alpha^2 (= \overline{5.8}^2) = 33.64$, and $\alpha^4 (= \overline{5.8}^4) = 1131.6496$, and $7\alpha^4 (= 7 \times 1131.6496) = 7921.5472$, and $155\alpha^2 (= 155 \times 33.64) = 1850.20$, and conſequently $10{,}000 + 7\alpha^4 + 155\alpha^2 (= 10{,}000 + 7921.5472 + 1850.20) = 19{,}771.7472$. Therefore we ſhall have $y^5 + 20y^3 (= 10{,}000 + 7\alpha^4 + 155\alpha^2) = 19{,}771.7472$; of which equation we muſt now endeavour to find the root.

Art. 32. Now we have ſeen above, that, if y be ſuppoſed to be $= 6$, the compound quantity $y^5 + 20y^3$ will be $= 12{,}096$; which is leſs than the abſolute term, 19,771.7472, of the preſent equation $y^5 + 20y^3 = 19{,}771.7472$. Therefore 6 muſt be leſs than the root of the ſaid equation.

We

$y^5 + 20y^3$ (= 100,000 + 20,000) = 120,000; which is much greater than the abſolute term 10,000. Therefore 10 muſt be much greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the 3d place, ſuppoſe y to be equal to 5. And then we ſhall have $y^3 = 125$, and $y^5 = 3125$, and $20y^3$ (= 20 × 125) = 2500, and $y^5 + 20y^3$ (= 3125 + 2500) = 5625; which is leſs than the abſolute term 10,000. Therefore 5 muſt be leſs than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the 4th place, ſuppoſe y to be equal to 6. And then we ſhall have $y^3 = 216$, and $y^5 = 7776$, and $20y^3$ (= 20 × 216) = 4320, and conſequently $y^5 + 20y^3$ (= 7776 + 4320) = 12,096; which is greater than the abſolute term 10,000. Therefore 6 muſt be greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$.

We will therefore, in the fifth place, ſuppoſe y to be equal to 5.8. And then we ſhall have $y^3 = 195.112$, and $y^5 = 6563.56768$, and $20y^3$ (= 20 × 195.112) = 3902.240, and conſequently $y^5 + 20y^3$ (= 6563.56768 + 3902.240) = 10,465.80768; which is ſomewhat greater than the abſolute term 10,000. Therefore 5.8 muſt be ſomewhat greater than the value of y in the equation $y^5 + 20y^3 = 10{,}000$. But the difference will be but ſmall, and conſequently we may conſider 5.8 as the root of the ſaid equation $y^5 + 20y^3 = 10{,}000$, and

as

as the firſt near value of a in the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$.

Art. 31. Let this quantity 5.8 be called α. And let us, for a ſecond near value of a, find the root of the equation $y^5 + 20y^3 = 10{,}000 + 7\alpha^4 + 155\alpha^2$. For this root will be greater than α, or 5.8, or the root of the equation $y^5 + 20y^3 = 10{,}000$, becauſe the trinomial quantity $10{,}000 + 7\alpha^4 + 155\alpha^2$ is greater than the ſingle quantity 10,000; but it will be leſs than a, or the root of the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$, becauſe the trinomial quantity $10{,}000 + 7\alpha^4 + 155\alpha^2$ is leſs than the trinomial quantity $10{,}000 + 7a^4 + 155aa$.

Now, ſince α is $= 5.8$, we ſhall have $\alpha^2 (= \overline{5.8}^2) = 33.64$, and $\alpha^4 (= \overline{5.8}^4) = 1131.6496$, and $7\alpha^4 (= 7 \times 1131.6496) = 7921.5472$, and $155\alpha^2 (= 155 \times 33.64) = 1850.20$, and conſequently $10{,}000 + 7\alpha^4 + 155\alpha^2 (= 10{,}000 + 7921.5472 + 1850.20) = 19{,}771.7472$. Therefore we ſhall have $y^5 + 20y^3 (= 10{,}000 + 7\alpha^4 + 155\alpha^2) = 19{,}771.7472$; of which equation we muſt now endeavour to find the root.

Art. 32. Now we have ſeen above, that, if y be ſuppoſed to be $= 6$, the compound quantity $y^5 + 20y^3$ will be $= 12{,}096$; which is leſs than the abſolute term, 19,771.7472, of the preſent equation $y^5 + 20y^3 = 19{,}771.7472$. Therefore 6 muſt be leſs than the root of the ſaid equation.

We

We will therefore ſuppoſe y to be equal to 7. And then we ſhall have $y^3 = 343$, and $y^5 = 16{,}807$, and conſequently $20y^3$ $(= 20 \times 343) = 6860$, and $y^5 + 20y^3$ $(= 16{,}807 + 6860) = 23{,}667$; which is greater than the abſolute term 19,771.7472. Therefore 7 muſt be greater than the root of the equation $y^5 + 20y^3 =$ 19,771.7472.

We will therefore, in the next place, ſuppoſe y to be equal to 6.7. And then we ſhall have $y^3 = 300.763$, and $y^5 = 13{,}501.25107$, and conſequently $20y^3$ $(= 20 \times 300.763) = 6015.260$, and $y^5 + 20y^3$ $(= 13{,}501.25107 + 6015.260) = 19{,}516.51107$; which is very little leſs than the abſolute term 19,771.7472. Therefore 6.7 muſt be very little leſs than the root of the equation $y^5 + 20y^3 = 19{,}771.7472$. We may therefore conſider 6.7 as the ſecond near value of a, or of the root of the equation $a^5 + 20a^3 = 10{,}000 + 7a^4 + 155aa$, or of the equation $a^5 - 7a^4 + 20a^3 - 155aa = 10{,}000$, or of the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$.

Art. 33. This ſecond near value of a, or of the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$, is conſiderably leſs than it's true value. For, if we ſuppoſe x to be equal only to this ſecond near value of a, to wit, 6.7, the two terms $x^5 + 20x^3$ will be leſs, inſtead of being greater, than the two terms $7x^4 + 155xx$, which are to be ſubtracted from them. For, upon this ſuppoſition we ſhall have $xx = 44.89$, and $x^3 = 300.763$, and $x^4 = 2015.1121$, and $x^5 = 13{,}501.25107$,

.25107, and $7x^4$ (= 7 × 2015.1121) = 14,105.7847, and $20x^3$ (= 20 × 300.763) = 6015.260, and $155xx$ (= 155 × 44.89) = 6957.95, and $x^5 + 20x^3$ (= 13,501.25107 + 6015.260) = 19,516.51107, and $7x^4 + 155xx$ (= 14,105.7847 + 6957.95) = 21,063.7347; which is greater than 19,516.51107, or $x^5 + 20x^3$.

Therefore x muſt be conſiderably greater than 6.7, in order to make $x^5 + 20x^3$ become, not only greater than $7x^4 + 155xx$, but greater by an exceſs equal to 10,000.

We will therefore ſuppoſe x to be equal to 8. And then we ſhall have $xx = 64$, and $x^3 = 512$, and $x^4 = 4096$, and $x^5 = 32768$, and conſequently $7x^4$ (= 7 × 4096) = 28672, and $20x^3$ (= 20 × 512) = 10240, and $155xx$ (= 155 × 64) = 9920, and $x^5 - 7x^4 + 20x^3 - 155xx$ (= 32768 − 28672 + 10240 − 9920 = 43,008 − 38,592) = 4416; which is leſs than 10,000, or the abſolute term of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. Therefore 8 will be ſomewhat leſs than the leaſt root of the ſaid equation.

We will therefore now ſuppoſe x to be = 8.5. And then we ſhall have $xx = 72.25$, and $x^3 = 614.125$, and $x^4 = 5220.0625$, and $x^5 = 44370.53125$, and conſequently $7x^4$ (= 7 × 5220.0625) = 36540.4375, and $20x^3$ (= 20 × 614.125) = 12282.500, and $155xx$ (= 155 × 72.25) = 11,198.75, and $x^5 - 7x^4 + 20x^3 - 155xx$ (= 44,370.53125 − 36,540.4375 + 12,282.500 − 11,198.75 = 56,653.03125 − 47,739.

.1875) = 8913.84375; which is leſs than the abſolute term 10,000. Therefore 8.5 will be leſs than the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$. But the difference between them will not be great, becauſe 8913.84375 is not a great deal leſs than 10,000. Therefore 8.5 will be ſufficiently near to the true value of the ſaid leaſt root to be the ground of a further approximation to it by Mr. Raphſon's method.

Art. 34. Let us therefore ſuppoſe x to be $= 8.5 + z$.

Then we ſhall have

$$xx \left(= \overline{8.5+z}^2 = \overline{8.5}^2 + 2 \times 8.5 \times z + \&c\right) = 72.25 + 17.0 \times z + \&c,$$

$$\text{and } x^3 \left(= \overline{8.5+z}^3 = \overline{8.5}^3 + 3 \times \overline{8.5}^2 \times z + \&c = 614.125 + 3 \times 72.25 \times z + \&c\right) = 614.125 + 216.75 \times z + \&c,$$

$$\text{and } x^4 \left(= \overline{8.5+z}^4 = \overline{8.5}^4 + 4 \times \overline{8.5}^3 \times z + \&c = 5220.0625 + 4 \times 614.125 \times z + \&c\right) = 5220.0625 + 2456.500 \times z + \&c,$$

$$\text{and } x^5 \left(= \overline{8.5+z}^5 = \overline{8.5}^5 + 5 \times \overline{8.5}^4 \times z + \&c = 44{,}370.53125 + 5 \times 5220.0625 \times z + \&c\right) = 44{,}370.53125 + 26{,}100.3125 \times z + \&c.$$

Therefore

Therefore $7x^4$ will be $(= 7 \times \overline{5220.0625 + 2456.500 \times z + \&c} = 7 \times 5220.0625 + 7 \times 2456.500 \times z + \&c) = 36540.4375 + 17195.500 \times z + \&c$, and $20x^3$ will be $(= 20 \times \overline{614.125 + 216.75 \times z + \&c} = 20 \times 614.125 + 20 \times 216.75 \times z + \&c) = 12,282.500 + 4335.00 \times z + \&c$, and $155xx$ will be $(= 155 \times \overline{72.25 + 17.0 \times z + \&c} = 155 \times 72.25 + 155 \times 17.0 \times z + \&c) = 11,198.75 + 2635.0 \times z + \&c$, and consequently $x^5 - 7x^4 + 20x^3 - 155xx$ will be $=$

$$\left\{\begin{array}{rrrr} 44,370.53125 & + 26,100.3125 & \times z & + \&c \\ - 36,540.4375 & - 17,195.500 & \times z & - \&c \\ + 12,282.500 & + 4335.00 & \times z & + \&c \\ - 11,198.75 & - 2635.0 & \times z & - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrrr} 56,653.03125 & + 30,435.3125 & \times z & + \&c \\ - 47,739.1875 & - 19,830.500 & \times z & - \&c \end{array}\right\}$$

$$= 8,913.84375 + 10,604.8125 \times z \quad \&c.$$

But $x^5 - 7x^4 + 20x^3 - 155xx$ is $= 10,000$.

Therefore $8,913.84375 + 10,604.8125 \times z + \&c$ will alſo be $= 10,000$.

And conſequently $10,604.8125 \times z$ will be $(= 10,000 - 8,913.84375) = 1086.15625$, and z will be $= \frac{1086.15625}{10,604.8125} = 0.10$.

Therefore 8.5 + z will be (= 8.5 + 0.10) = 8.60, or 8.6, and consequently x, or the least root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$ will be, nearly, = 8.60, or 8.6. Q. E. I.

Art. 35. Now let 8.6 be substituted instead of x in the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$. And we shall have xx (= $\overline{8.6}^2$) = 73.96, and x^3 = 636.056, and x^4 = 5470.0816, and x^5 = 47,042.701,76, and consequently $7x^4$ (= 7 × 5470.0816) = 38,290.5712, and $20x^3$ (= 20 × 636.056) = 12,721.120, and $155xx$ (= 155 × 73.96) = 11,463.80, and $x^5 - 7x^4 + 20x^3 - 155xx$ (= 47,042.701,76 — 38,290.5712 + 12,721.120 — 11,463.80 = 59,763.821,76 — 49,754.3712) = 10,009.450,56; which is a little greater than the absolute term 10,000. Therefore 8.6 must be a little greater than the true value of the least root of the proposed equation $x^5 - 7x^4 + 20x^3 - 155xx$ = 10,000.

Art. 36. We have now seen that, when x is = 8.5, the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will be equal to 8913.84375; and that, when x is = 8.6, the said compound quantity will be equal to 10,009.45056. Therefore, while x increases from 8.5 to 8.6, the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will increase from 8913.84375 to 10,009.45056. Therefore there will be some point of time, during the increase of x from 8.5 to 8.6, at which the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ will be equal to 10,000, which is of an intermediate

intermediate magnitude between 8913.84375 and 10,009.45056; or, in other words, the root of the equation $x^5 - 7x^4 + 20x^3 - 155xx$ will be of an intermediate magnitude between 8.5 and 8.6. But it will be much nearer to 8.6 than to 8.5, becauſe 10,009.45056 is much nearer than 8913.84375 to 10,000.

We will therefore ſuppoſe x to be $= 8.6 - v$.

And then we ſhall have

$xx\ (= \overline{8.6-v}^2 = \overline{8.6}^2 - 2 \times 8.6 \times v + \&c) =$
$73.96 - 17.2 \times v + \&c,$

and $x^3\ (= \overline{8.6-v}^3 = \overline{8.6}^3 - 3 \times \overline{8.6}^2 \times v + \&c$
$= 636.056 - 3 \times 73.96 \times v + \&c)$
$= 636.056 - 221.88 \times v + \&c,$

and $x^4\ (= \overline{8.6-v}^4 = \overline{8.6}^4 - 4 \times \overline{8.6}^3 \times v + \&c$
$= 5470.0816 - 4 \times 636.056 \times v + \&c)$
$= 5470.0816 - 2544.224 \times v + \&c,$

and $x^5\ (= \overline{8.6-v}^5 = \overline{8.6}^5 - 5 \times \overline{8.6}^4 \times v + \&c$
$= 47,042.70176 - 5 \times 5470.0816 \times$
$v + \&c) = 47,042.70176 - 28350.4080$
$\times v + \&c,$

and conſequently $7x^4\ (= 7 \times$
$\overline{5470.0816 - 2544.224 \times v + \&c} = 7 \times$
$5470.0816 - 7 \times 2544.224 \times v + \&c)$
$= 38,290\ 5712 - 17,809.568 \times v + \&c,$

and $20x^3$ $(= 20 \times \overline{636.056 - 221.88 \times v + \&c} =$
$20 \times 636.056 - 20 \times 221.88 \times v + \&c)$
$12,721.120 - 4437.60 \times v + \&c,$

and $155xx$ $(= 155 \times \overline{73.96 - 17.2 \times v \times \&c} =$
$155 \times 73.96 - 155 \times 17.2 \times v + \&c)$
$= 11,463.80 - 2666.0 \times v + \&c,$

and $x^5 - 7x^4 + 20x^3 - 155xx =$

$$\left\{\begin{array}{rrrl} 47,042.70176 & - 28,350.4080 & \times v + & \&c \\ - 38,296.5712 & + 17,809.568 & \times v - & \&c \\ + 12,721.120 & - 4,437.60 & \times v + & \&c \\ - 11,463.80 & + 2,666.0 & \times v - & \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrrl} 59,763.821,76 & - 32,788.0080 & \times v + & \&c \\ - 49,754.371,2 & + 20,475.568 & \times v - & \&c \end{array}\right\}$$

$$= 10,009.450,56 - 12,312.4400 \times v \quad \&c.$$

But $x5 - 7x^4 + 20x^3 - 155xx$ is $= 10,000.$

Therefore $10,009.450,56 - 12,312.4400 \times v$ will likewiſe be $= 10,000.$

And conſequently $10,009.450,56$ will be $= 10,000 + 12,312.4400 \times v$, and $12,312.4400 \times v$ will be $(= 10,009.450,56 - 10,000) = 9.450,56$, and v will be $=$

$$\frac{9.450,56}{12,312.4400} = 0.000,767,5.$$

Therefore

Therefore 8.6 — v will be (= 8.6 — 0.000,767,5) = 8.599,232,5; and consequently x, or the leaſt root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$, will be = 8.599,232,5 Q. E. I.

Art. 37. This number 8.599,232,5 is exact in either all it's figures, or all but the laſt figure 5. But the trial of it by ſubſtituting it inſtead of x in the compound quantity $x^5 - 7x^4 + 20x^3 - 155xx$ would be attended with a good deal of labour. However, it may be performed with a tolerable degree of exactneſs, though not with perfect exactneſs, by the help of logarithms, without any great trouble, in the manner following:

The logarithm of 8.599,200,0 is 0.934,458,0, and the logarithm of 8,599,300,0 is 0.934,463,1, which exceeds the former logarithm by 0.000,005,1. Therefore the logarithm of 8.599,232,5 will be of an intermediate magnitude between 0.934,458,0 and 0.934,463,1, and its exceſs above 0.934,458,0 is to be found by the following proportion, to wit, As 8.599,300,0 — 8.599,200,0 is to 8.599,232,5 — 8.599,200,0, ſo is 0.934,463,1 — 0.934,458,0 to the exceſs of the logarithm of 8.599,232,5 above 0.934,458,0, that is, as 0.000,100,0 is to 0.000,032,5, ſo is 0.000,005,1 to the ſaid exceſs, or as 1000 to 325, ſo is 0.000,005,1 to the ſaid exceſs. Therefore the ſaid exceſs will be $= \frac{325 \times 0.000{,}005{,}1}{1000} = \frac{0.001{,}657{,}5}{1000}$ = 0.000,001,6; and conſequently the logarithm of 8.599,232,5 will be (= 0.934,458,0 + 0.000,001,6)

$= 0.934,459,6$. Therefore, if x be ſuppoſed to be equal to 8.599,232,5, the logarithm of xx will be $(= 2 \times 0.934,459,6) = 1.868,919,2$, and the logarithm of x^3 will be $(= 3 \times 0.934,459,6) = 2.803,378,8$, and the logarithm of x^4 will be $(= 4 \times 0.934,459,6) = 3.737,838,4$, and the logarithm of x^5 will be $(= 5 \times 0.934,459,6) = 4.672,298,0$. Therefore xx will be $= 73.946,762,7$, and x^3 will be $= 635.885,294,1$; and x^4 will be $= 5468.125,000,0$, and x^5 will be $= 47,021.663,043,4$. Therefore $7x^4$ will be $(= 7 \times 5468.125,000,0) = 38,276.875,000,0$, and $20x^3$ will be $(= 20 \times 635.885,294,1) = 12,717.705,882,0$, and $155xx$ will be $(= 155 \times 73.946,762,7) = 11,461.748,218,5$, and $x^5 - 7x^4 + 20x^3 - 155xx$ will be $(= 47,021.663,043,4 - 38,276.875,000,0 + 12,717.705,882,0 - 11,461.748,218,5 = 59,739.368,925,4 - 49,738.623,218,5) = 10,000.745,707,9$; which exceeds 10,000, or the abſolute term of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$, by leſs than an unit, or the 10,000th part of the ſaid abſolute term. Therefore 8.599,232,5 muſt be very nearly equal to the root of the ſaid equation. Q. E. D.

Art. 38. This number 8.599,232,5 is the only root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10,000$. For, if we ſhould ſuppoſe it to have any other root greater than a, or 8.599,232,5, an impoſſible concluſion would follow from ſuch a ſuppoſition; as may be ſhewn in the manner following:

Let

Let b denote the other and greater root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$, if it can have ſuch other root.

Then will the compound quantity $b^5 - 7b^4 + 20b^3 - 155bb$ be $= 10{,}000$; and conſequently it will alſo be equal to $a^5 - 7a^4 + 20a^3 - 155aa$. Therefore (adding $7b^4$ and $155bb$ to both ſides,) we ſhall have $b^5 + 20b^3 = a^5 + 7b^4 - 7a^4 + 20a^3 + 155bb - 155aa$; and (ſubtracting $a^5 + 20a^3$ from both ſides) we ſhall have $b^5 - a^5 + 20b^3 - 20a^3 = 7b^4 - 7a^4 + 155bb - 155aa$, or $b^5 - a^5 + 20 \times \overline{b^3 - a^3} = 7 \times \overline{b^4 - a^4} + 155 \times \overline{bb - aa}$, and conſequently (dividing both ſides of the equation by $b - a$,) $b^4 + b^3a + b^2a^2 + ba^3 + a^4 + 20 \times \overline{bb + ba + aa} = 7 \times \overline{b^3 + ba + ba^2 + a^3} + 155 \times \overline{b + a}$, or $b^4 + b^3a + b^2a^2 + ba^3 + a^4 + 20bb + 20ba + 20aa = 7b^3 + 7b^2a + 7ba^2 + 7a^3 + 155b + 155a$. Therefore, if we ſubtract $7a^3 + 155a$ from both ſides of this equation, we ſhall have $7b^3 + 7b^2a + 7ba^2 + 155b = b^4 + b^3a + b^2a^2 + ba^3 + 20bb + 20ba + a^4 - 7a^3 + 20aa - 155a$; and, if we ſubtract $b^4 + b^3a + b^2a^2 + ba^3 + 20bb + 20ba$ from both ſides, we ſhall have $7b^3 + 7b^2a + 7ba^2 + 155b - b^4 - b^3a - b^2a^2 - ba^3 - 20bb - 20ba = a^4 - 7a^3 + 20aa - 155a$, or

$$\left.\begin{array}{l} 155b - 20bb + 7b^3 - b^4 \\ - 20ab + 7abb - ab^3 \\ + 7a^2b - a^2bb \\ - a^3b \end{array}\right\} =$$

$= a^4 - 7a^3 + 20aa - 155a = \frac{a^5 - 7a^4 + 20aa + 155aa}{a}$

$= \frac{10,000}{a}$, or (if we ſubſtitute 8.6 for a, to which it is very nearly equal,)

$$\left\{\begin{array}{l} 155b - 20bb + 7b^3 - b^4 \\ -\ 20 \times 8.6b + 7 \times 8.6bb - 8.6b^3 \\ +\ 7 \times 73.96 \times b - 73.96bb \\ -\ 636.056 \times b \end{array}\right\}$$

$= \frac{10,000}{8.6} = 1162.7$, or

$$\left\{\begin{array}{l} 155b - 20bb + 7 \times b^3 - b^4 \\ -\ 172.0 \times b + 60.2bb - 8.6 \times b^3 \\ +\ 517.72 \times b - 73.96bb \\ -\ 636.056 \times b \end{array}\right\}$$

$= 1162.7$, or

$$\left\{\begin{array}{l} 672.72 \times b - 93.96bb + 7b^3 - b^4 \\ -\ 808.056 \times b + 60.2\ bb - 8.6 \times b^3 \end{array}\right\}$$

$= 1162.7$; that is, if from the three terms $672.72 \times b + 60.2 \times bb + 7b^3$ we ſubtract the four terms $808.056 \times b + 93.96 \times bb + 8.6 \times b^3 + b^4$, the remainder will be equal to 1162.7. But the ſaid four terms $808.056 \times b + 93.96 \times bb + 8.6 \times b^3 + b^4$, are greater than the ſaid three terms $672.72 \times b + 60.2 \times bb + 7b^3$, and therefore cannot be ſubtracted from them. Therefore the ſuppoſition, from which it followed that

that the ſaid four terms would be leſs than the ſaid three terms, and might be ſubtracted from them, and that the remainder would be equal to 1162.7, or the ſuppoſition that the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$ had a ſecond real and affirmative root, b, that was greater than a, or 8.6, could not be a true ſuppoſition. We may therefore conclude that 8.6, or 8.599,232,5, is the only root of the propoſed equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$. Q. E. D.

Another Example of the Reſolution of an Equation that, by the Form of it, or the Changes of the Signs + and − prefixed to it's Terms, ſeems capable of having more than One real and affirmative Root, by means of the foregoing Proceſſes deſcribed above in Art. 9, 10, 11, 12, *and* 13.

Art. 39. Let it be required to reſolve the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x = 5$, (which riſes to the eighth power of x,) by means of the ſaid proceſſes.

This equation is taken from Sir Iſaac Newton's *Arithmetica Univerſalis*, page 276 of the ſecond edition, and is the higheſt numeral equation mentioned in that learned treatiſe.

$y^8 + 4y^7 + 5y^4$ ($= 1 + 4 + 5$) $= 10$; which is greater than the abſolute term 5. Therefore 1 muſt be greater than the true value of y in the equation $y^8 + 4y^7 + 5y^4 = 5$.

We will therefore, in the ſecond place, ſuppoſe y to be $= 0.7$.

And then we ſhall have $yy = 0.49$, and $y^3 = 0.343$, and $y^4 = 0.2401$, and $y^5 = 0.168,07$, and $y^6 = 0.117,649$, and $y^7 = 0.082,354,3$, and $y^8 = 0.057,648,01$, and conſequently $4y^7$ ($= 4 \times 0.082,354,3$) $= 0.329,417,2$, and $5y^4$ ($= 5 \times 0.2401$) $= 1.2005$, and $y^8 + 4y^7 + 5y^4$ ($= 0.057,648,01 + 0.329,417,2 + 1.2005$) $= 1.587,565,31$; which is leſs than the abſolute term 5. Therefore 0.7 muſt be leſs than the value of y in the equation $y^8 + 4y^7 + 5y^4 = 5$.

We will therefore, in the 3d place, ſuppoſe y to be $= 0.9$.

And then we ſhall have $yy = 0.81$, and $y^3 = 0.729$, and $y^4 = 0.6561$, and $y^5 = 0.59049$, and $y^6 = 0.531,441$, and $y^7 = 0.478,296,9$, and $y^8 = 0.430,467,21$, and conſequently $4y^7$ ($= 4 \times 0.478,296,9$) $= 1.913,187,6$, and $5y^4$ ($= 5 \times 0.6561$) $= 3.2800$, and $y^8 + 4y^7 + 5y^4$ ($= 0.430,467,21 + 1.913,187,6 + 3.2800$) $= 5.623,654,81$; which is ſomewhat greater than the abſolute term 5. Therefore 0.9 muſt be ſomewhat greater than the root of the equation $y^8 + 4y^7 + 5y^4$

$= 5$. But the difference will not be great; and conſequently we may conſider this quantity 0.9 as the firſt near value of a, or of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, of which we are in ſearch.

Art. 42. Now let α be put for 0.9, or the ſaid firſt near value of a. And, as a ſecond near value of it, let us ſeek the root of the equation $y^8 + 4y^7 + 5y^4 = 5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$; which will be greater than α, but leſs than a, becauſe the ſextinomial quantity $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ is greater than the ſingle quantity 5, but leſs than the ſextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$.

Now, becauſe α is $= 0.9$, we ſhall have α^6 $(= \overline{0.9}^6)$ $= 0.531,441$, and $10\alpha^5$ $(= 10 \times 0.59049) = 5.9049$, and $5\alpha^3$ $(= 5 \times 0.729) = 3.645$, and $10\alpha^2$ $(= 10 \times 0.81)$ $= 8.1$, and 10α $(= 10 \times 0.9) = 9$, and conſequently $5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ $(= 5 + 0.531,441 + 5.9049 + 3.645 + 8.1 + 9) = 32.181,341$; and therefore the equation $y^8 + 4y^7 + 5y^4 = 5 + \alpha^6 + 10\alpha^5 + 5\alpha^3 + 10\alpha^2 + 10\alpha$ will be $y^8 + 4y^7 + 5y^4 = 32.181,341$, or (neglecting the decimal fraction 0.181, 341,) $y^8 + 4y^7 + 5y^4 = 32$. Of this equation we are now to find the root.

Art. 43. Now let y (which we know to be greater than α, or 0.9,) be ſuppoſed to be $= 1.2$.

Then

Then we ſhall have $yy = 1.44$, and $y^3 = 1.728$, and $y^4 = 2.0736$, and $y^5 = 2.488,32$, and $y^6 = 2.985,984$, and $y^7 = 3.583,180,8$, and $y^8 = 4.299,816,96$, and conſequently $4y^7$ ($= 4 \times 3.583,180,8$) $= 14.332,723,2$, and $5y^4$ ($= 5 \times 2.0736$) $= 10.3680$, and $y^8 + 4y^7 + 5y^4$ ($= 4.299,816,96 + 14.332,723,2 + 10.3680$) $= 29.000,540,16$; which is ſomewhat leſs than the abſolute term 32. Therefore 1.2 is ſomewhat leſs than the root of the equation $y^8 + 4y^7 + 5y^4 = 32$. But the difference between them is not great. And therefore we may conſider 1.2 as the ſecond near value of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, of which we are in ſearch.

Art. 44. Now let $\mathfrak{C}$ be put for 1.2, or this ſecond near value of the root a of the ſaid equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$. And for a third near value of it, let us ſeek the root of the equation $y^8 + 4y^7 + 5y^4 = 5 + \mathfrak{C}^6 + 10\mathfrak{C}^5 + 5\mathfrak{C}^3 + 10\mathfrak{C}^2 + 10\mathfrak{C}$; which will be greater than $\mathfrak{C}$, or the root of the equation $y^8 + 4y^7 + 5y^4 = 32$, or $y^8 + 4y^7 + 5y^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, but leſs than a, or the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, becauſe the ſextinomial quantity $5 + \mathfrak{C}^6 + 10\mathfrak{C}^5 + 5\mathfrak{C}^3 + 10\mathfrak{C}^2 + 10\mathfrak{C}$ is greater than the ſextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, but leſs than the ſextinomial quantity $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$.

Now, becauſe $\mathfrak{C}$ is $= 1.2$, we ſhall have $\mathfrak{C}^6$ ($= \overline{1.2}^6$) $= 2.985,984$, and $10\mathfrak{C}^5$ ($= 10 \times \overline{1.2}^5 = 10 \times 2.488,32$)

2.488,32) = 24.8832, and $5c^3$ (= 5 × $\overline{1.2}|^3$ = 5 × 1.728) = 8.640, and $10c^2$ (= 10 × $\overline{1.2}|^2$ = 10 × 1.44) = 14.4, and 10c (= 10 × 1.2) = 12, and consequently $5 + c^5 + 10c^4 + 5c^3 + 10c^2 + 10c$ (= 5 + 2.985,984 + 24.8832 + 8.640 + 14.4 + 12) = 67.909,184. Therefore the equation $y^5 + 4y^7 + 5y^4 = 5 + c^5 + 10c^4 + 5c^3 + 10c^2 + 10c$ will be $y^5 + 4y^7 + 5y^4 = 67.909,184$, or, nearly, $y^5 + 4y^7 + 5y^4 = 68$. This equation we muſt now endeavour to reſolve.

Art. 45. Now let y (which we know to be greater than c, or 1.2,) be ſuppoſed to be = 1.3. Then we ſhall have yy (= $\overline{1.3}|^2$) = 1.69, and y^3 (= $\overline{1.3}|^3$) = 2.197, and y^4 = 2.8561, and y^5 = 3.712,93, and y^6 = 4.826,809, and y^7 = 6.274,851,7, and y^8 = 8.157,306,82, and conſequently $4y^7$ (= 4 × 6.274,851,7) = 26.099,406,8, and $5y^4$ (= 5 × 2.8561) = 14.2805, and $y^8 + 4y^7 + 5y^4$ (= 8.157,306,82 + 26.099,406,8 + 14.2805) = 48.537,213,62; which is leſs than the abſolute term 68. Therefore 1.3 will be leſs than the root of the equation $y^8 + 4y^7 + 5y^4 = 68$.

Now let y be ſuppoſed to be = 1.4.

And then we ſhall have yy = 1.96, and y^3 = 2.744, and y^4 = 3.8416, and y^5 = 5.37824, and y^6 = 7.529,536, and y^7 = 10.541,350,4, and y^8 = 14.757,890,56, and conſequently $4y^7$ (= 4 × 10.541,350,4) = 42.165,401,6, and $5y^4$ (= 5 × 3.8416) = 19.2080, and y^8 +

$4y^7$

$4y^7 + 5y^4$ ($= 14.757,890,56 + 42.165,401,6 + 19.2080$) $= 76.131,292,16$; which is greater than the abſolute term 68. Therefore 1.4 will be greater than the root of the equation $y^8 + 7y^7 + 5y^4 = 68$. But the difference will not be great. And therefore we may conſider 1.4 as a third near value of the root of the equation $a^8 + 4a^7 + 5a^4 = 5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, or as a third near value of the leaſt root (if it has more than one root,) of the equation $x^8 + 4x^7 + 5x^4 = 5 + x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$, or of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

Art. 46. Now let 1.4 be ſubſtituted inſtead of x in the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$; and it will be $= 14.757,890,56 + 4 \times 10.541,350,4 - 7.529,536 - 10 \times 5.37824 + 5 \times 3.8416 - 5 \times 2.744 - 10 \times 1.96 - 10 \times 1.4 = 14.757,890,56 + 42.165,401,6 - 7.529,536 - 53.7824 + 19.2080 - 13.720 - 19.6 - 14 = 76.131,292,16 - 108.631,936$; by which it appears that the three terms $x^8 + 4x^7 + 5x^4$, which ought to be greater than all the five terms $x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$, and to exceed them by the number 5, will be leſs than the ſaid five terms. Therefore 1.4 will be leſs than the value of x in the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

Art. 47. We will therefore ſuppoſe x to be $= 1.6$.

And then we ſhall have $xx = 2.56$, and $x^3 = 4.096$, and $x^4 = 6.5536$, and $x^5 = 10.48576$, and $x^6 = 16.777,216$, and $x^7 = 26.843,545,6$, and $x^8 = 42.949,672,96$, and conſequently $4x^7$ $(= 4 \times 26.843,545,6) = 107.374,182,4$, and $10x^5$ $(= 10 \times 10.48576) = 104.8576$, and $5x^4$ $(= 5 \times 6.5536) = 32.7680$, and $5x^3$ $(= 5 \times 4.096) = 20.480$, and $10x^2$ $(= 10 \times 2.56) = 25.6$, and $10x$ $(= 10 \times 1.6) = 16$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ $(= 42.949,672,96 + 107.374,182,4 - 16.777,216 - 104.8576 + 32.7680 - 20.480 - 25.6 - 16) = 183.091,855,36 - 183.714,816$; of which quantities the latter, to wit, 183.714,816, (which is equal to $x^6 + 10x^5 + 5x^3 + 10x^2 + 10x$,) is greater than the former, to wit, 183.091,855,36, (which is equal to $x^8 + 4x^7 + 5x^4$,) and therefore cannot be ſubtracted from it, as it is ſuppoſed to be in the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. Therefore the root of that equation muſt be ſomewhat greater than 1.6.

Art. 48. Let us therefore ſuppoſe x to be $= 1.6 + z$, and let us proceed to inveſtigate the value of z by Mr. Raphſon's method of approximation.

Then, ſince x is $= 1.6 + z$, we ſhall have

$$xx \left(= \overline{1.6+z}^2 = \overline{1.6}^2 + 2 \times 1.6 \times z + \&c\right) = 2.56 + 3.2 \times z + \&c,$$

and

$$x^3 \left(= \overline{1.6+z}^3 = \overline{1.6}^3 + 3 \times \overline{1.6}^2 \times z + \&c = \overline{1.6}^3 + 3 \times 2.56 \times z + \&c\right) = 4.096 + 7.68 \times z + \&c,$$

and

and $x^4 (= \overline{1.6+z}^4 = \overline{1.6}^4 + 4 \times \overline{1.6}^3 \times z + \&c$
$= \overline{1.6}^4 + 4 \times 4.096 \times z + \&c) =$
$6.5536 + 16.384 \times z + \&c,$

and $x^5 (= \overline{1.6+z}^5 = \overline{1.6}^5 + 5 \times \overline{1.6}^4 \times z + \&c$
$= \overline{1.6}^5 + 5 \times 6.5536 \times z + \&c) =$
$10.48576 + 32.7680 \times z + \&c,$

and $x^6 (= \overline{1.6+z}^6 = \overline{1.6}^6 + 6 \times \overline{1.6}^5 \times z +$
$\&c = \overline{1.6}^6 + 6 \times 10.48576 \times z + \&c)$
$= 16.777,216 + 62.91456 \times z + \&c,$

and $x^7 (= \overline{1.6+z}^7 = \overline{1.6}^7 + 7 \times \overline{1.6}^6 \times z + \&c$
$= \overline{1.6}^7 + 7 \times 16.777,216 \times z + \&c)$
$= 26.843,545,6 + 117.440,512 \times z$
$+ \&c,$

and $x^8 (= \overline{1.6+z}^8 = \overline{1.6}^8 + 8 \times \overline{1.6}^7 \times z + \&c$
$= \overline{1.6}^8 + 8 \times 26.843,545,6 \times z + \&c)$
$= 42.949,672,96 + 214.748,364,8 \times z$
$+ \&c,$

and consequently $4x^7 (= 4 \times$
$26.843,545,6 + 117.440,512 \times z + \&c$
$= 4 \times 26.843,545,6 + 4 \times 117.440,512$
$\times z + \&c) = 107.374,182,4 + 469.$
$762,048 \times z + \&c,$

and $10x^5$ $(= 10 \times \overline{10.48576 + 32.7680 \times z + \&c}$
$= 10 \times 10.48576 + 10 \times 32.7680$
$\times z + \&c) = 104.8576 + 327.680$
$\times z + \&c,$

and $5x^4$ $(= 5 \times \overline{6.5536 + 16.384 \times z + \&c} = 5 \times$
$6.5536 + 5 \times 16.384 \times z + \&c) =$
$32.7680 + 81.920 \times z + \&c,$

and $5x^3$ $(= 5 \times \overline{4.096 + 7.68 \times z + \&c} = 5 \times$
$4.096 + 5 \times 7.68 \times z + \&c) =$
$20.480 + 38.40 \times z + \&c,$

and $10x^2$ $(= 10 \times \overline{2.56 + 3.2 \times z + \&c} = 10 \times$
$2.56 + 10 \times 3.2 \times z + \&c) = 25.6$
$+ 32 \times z + \&c,$

and $10x$ $(= 10 \times \overline{1.6 + z} = 10 \times 1.6 + 10 \times z)$
$= 16 + 10z.$

Therefore the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ will be equal the complicated quantity

$$\left\{\begin{array}{rrrl} 42.949,672,96 & + 214.748,364,8 & \times z + \&c \\ + 107.374,182,4 & + 469.762,048 & \times z + \&c \\ - 16.777,216, & - 62.914,56 & \times z - \&c \\ - 104.857,6 & - 327.680 & \times z - \&c \\ + 32.768,0 & + 81.920 & \times z + \&c \\ - 20.480 & - 38.40 & \times z - \&c \\ - 25.6 & - 32 & \times z - \&c \\ - 16. & - 10 & \times z \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 183.091,855,36 & + 766,430,412,8 & \times z + \&c \\ -183.714,816,00 & - 470.994,560,0 & \times z - \&c \end{array}\right\}$$

$$= - 0.622,960,64 + 295.435,852,8 \times z \&c.$$

But the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ is $= 5$.

Therefore $295.435,852,8 \times z$ &c $- 0.622,960,64$ will alſo be $= 5$. And conſequently (adding 0.622,960,64 to both ſides,) we ſhall have $295.435,852,8 \times z (= 5 + 0.622,960,64) = 5.622,960,64$, and $z = \frac{5.622,960,64}{295.435,852,8} = 0.019$. Therefore $1.6 + z$, or x, will be $(= 1.6 + 0.019) = 1.619$; that is, the leaſt root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ will be 1.619.

Q. E. I.

Art. 49. This value of x will be pretty near the truth, but will exceed it by a very ſmall quantity. For, if we

ſuppoſe x to be $= 1.619$, we ſhall have $xx = 2.621,161$, and $x^3 = 4.243,659$, and $x^4 = 6.870,484$, and $x^5 = 11.123,213$, and $x^6 = 18.008,482$, and $x^7 = 29.155,732$, and $x^8 = 47.203,130$, and conſequently $4x^7$ $(= 4 \times 29.155,732) = 116.622,928$, and $10x^5$ $(= 10 \times 11.123,213) = 111.232,130$, and $5x^4$ $(= 5 \times 6.870,484) = 34.352,420$, and $5x^3$ $(= 5 \times 4.243,659) = 21.218,295$, and $10x^2$ $(= 10 \times 2.621,161) = 26.211,610$, and $10x$ $(= 10 \times 1.619) = 16.190,000$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x =$

$$\left\{\begin{array}{rrr} & 47.203,130 & \\ + & 116,622,928 & - \quad 18.008,482 \\ & & - \quad 111.232,130 \\ + & 34.352,420 & - \quad 21.218,295 \\ & & - \quad 26.211,610 \\ & & - \quad 16.190,000 \end{array}\right\} =$$

$198.178,478 - 192.860,517 = 5.317,961$; which is ſomewhat greater than 5, or the abſolute term of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. Therefore 1.619 will be ſomewhat greater than the true value of x in that equation. Q. E. D.

Art. 50. We will therefore, in the next place, ſuppoſe x to be $= 1.619 - v$.

And then we ſhall have

xx (=

xx $(= \overline{1.619 - v}^2 = \overline{1.619}^2 - 2 \times 1.619 \times v + \&c = 2.621,161 - 3.238 \times v + \&c,$

and x^3 $(= \overline{1.619 - v}^3 = \overline{1.619}^3 - 3 \times \overline{1.619}^2 \times v + \&c = \overline{1.619}^3 - 3 \times 2.621,161 \times v + \&c) = 4.243,659 - 7.863,483 \times v + \&c,$

and x^4 $(= \overline{1.619 - v}^4 = \overline{1.619}^4 - 4 \times \overline{1.619}^3 \times v + \&c = \overline{1.619}^4 - 4 \times 4.243,659 \times v + \&c) = 6.870,484 - 16.974,636 \times v + \&c,$

and x^5 $(= \overline{1.619 - v}^5 = \overline{1.619}^5 - 5 \times \overline{1.619}^4 \times v + \&c = \overline{1.619}^5 - 5 \times 6.870,484 \times v + \&c) = 11.123,213 - 34.352,420 \times v + \&c,$

and x^6 $(= \overline{1.619 - v}^6 = \overline{1.619}^6 - 6 \times \overline{1.619}^5 \times v + \&c = \overline{1.619}^6 - 6 \times 11.123,213 \times v + \&c) = 18.008,482 - 66.739,278 \times v + \&c,$

and x^7 $(= \overline{1.619 - v}^7 = \overline{1.619}^7 - 7 \times \overline{1.619}^6 \times v + \&c = \overline{1.619}^7 - 7 \times 18,008,482 \times v + \&c) = 29.155,732 - 126.059,374 \times v + \&c,$

and x^8 $(= \overline{1.619 - v}^8 = \overline{1.619}^8 - 8 \times \overline{1.619}^7 \times v + \&c = \overline{1.619}^8 - 8 \times 29.155,732 \times v + \&c) = 47.203,130 - 233.245,856 \times v + \&c,$

and consequently $4x^7$ $(= 4 \times 29.155,732 - 4 \times 126.059,374 \times v + \&c) = 116.622,928 - 504.237,496 \times v + \&c,$

and $10x^6$ $(= 10 \times 11.123,213 - 10 \times 34.352,420 \times v + \&c) = 111.232,130 - 343.524,200 \times v + \&c,$

and $5x^4$ $(= 5 \times 6.870,484 - 5 \times 16.974,636 \times v + \&c) = 34.352,420 - 84.873,180 \times v + \&c,$

and $5x^3$ $(= 5 \times 4.243,659 - 5 \times 7.863,483 \times v + \&c) = 21.218,295 - 39.317,415 \times v + \&c,$

and $10x^2$ $(= 10 \times 2.621,161 - 10 \times 3.238 \times v + \&c) = 26.211,610 - 32.380,000 \times v + \&c,$

and $10x$ $(= 10 \times 1.619 - 10 \times v) = 16.190,000 - 10.000,000 \times v,$

and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x =$

47.203,

$$\left\{\begin{array}{l} 47.203,130 - 233.245,856 \times v + \&c \\ + 116.622,928 - 504.237,496 \times v + \&c \\ - 18.008,482 + 66.739,278 \times v - \&c \\ - 111.232,130 + 343.524,200 \times v - \&c \\ + 34.352,420 - 84.873,180 \times v + \&c \\ - 21.218,295 + 39.317,415 \times v - \&c \\ - 26.211,610 + 32.380,090 \times v - \&c \\ - 16.190,000 + 10.000,000 \times v \end{array}\right\}$$

$$= \left\{\begin{array}{l} 198.178,478 - 822.356,532 \times v + \&c \\ - 192.860,517 + 491.960,893 \times v - \&c \end{array}\right\}$$

$$= 5.317,961 - 330.395,639 \times v \quad \&c.$$

But the compound quantity $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ is $= 5$.

Therefore $5.317,961 - 330.395,639 \times v$ will alſo be $= 5$.

And conſequently $5.317,961$ will be $= 5 + 330.395,639 \times v$, and $0.317,961$ will be $= 330.395,639 \times v$. Therefore v will be $= \frac{0.317,961}{330.395,639} = 0.000,962,3$, and $1.619 - v$ will be $(= 1.619,000,0 - 0.000,962,3) = 1.618,037,7$; that is, x, or the leaſt root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ will be nearly equal to $1.618,037,7$. Q. E. I.

This

This value, 1.618,037,7, of the leaſt root of the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 + 10x = 5$ is, probably, exact in the firſt ſeven places of figures 1,618,037.

Art. 51. In order to diſcover whether the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 + 10x = 5$ has any other root greater than 1.618,037, we may proceed as follows:

Let the root already found, to wit, 1.618,037,7, or (neglecting the three laſt figures 377,) 1.618, be called a, as before; and the greater root of the equation (if there be any ſuch root,) be called b.

Then we ſhall have $b^8 + 4b^7 - b^6 - 10b^5 + 5b^4 - 5b^3 - 10b^2 - 10b = 5$, and conſequently $= a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10a^2 - 10a$. Therefore (adding $b^6 + 10b^5 + 5b^3 + 10b^2 + 10b$ to both ſides of the equation,) we ſhall have $b^8 + 4b^7 + 5b^4 = a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10a^2 - 10a + b^6 + 10b^5 + 5b^3 + 10b^2 + 10b = a^8 + 4a^7 + b^6 - a^6 + 10b^5 - 10a^5 + 5a^4 + 5b^3 - 5a^3 + 10b^2 - 10a^2 + 10b - 10a$, and (ſubtracting $a^8 + 4a^7 + 5a^4$ from both ſides,) $b^8 - a^8 + 4b^7 - 4a^7 + 5b^4 - 5a^4 = b^6 - a^6 + 10b^5 - 10a^5 + 5b^3 - 5a^3 + 10b^2 - 10a^2 + 10b - 10a$, or $b^8 - a^8 + 4 \times \overline{b^7 - a^7} + 5 \times \overline{b^4 - a^4} = b^6 - a^6 + 10 \times \overline{b^5 - a^5} + 5 \times \overline{b^3 - a^3} + 10 \times \overline{b^2 - a^2} + 10 \times \overline{b - a}$, and (dividing both ſides of the equation by $b - a$,) we ſhall

ſhall have $\frac{b^8 - a^8}{b - a} + 4 \times \frac{b^7 - a^7}{b - a} + 5 \times \frac{b^4 - a^4}{b - a} =$

$\frac{b^6 - a^6}{b - a} + 10 \times \frac{b^5 - a^5}{b - a} + 5 \times \frac{b^3 - a^3}{b - a} + 10 \times$

$\frac{b^2 - a^2}{b - a} + 10 \times \frac{b - a}{b - a}$, or $b^7 + b^6a + b^5a^2 + b^4a^3$

$+ b^3a^4 + b^2a^5 + ba^6 + a^7 + 4 \times$

$\overline{b^6 + b^5a + b^4a + b3a^3 + b\,a^4 + ba^5 + a^6} + 5 \times$

$\overline{b^3 + b^2a + ba^2 + a^3} = b^5 + b^4a + b^3a^2 + b^2a3 + ba^4$

$+ a^5 + 10 \times \overline{b4 + b3a + b^2a + ba^3 + a^4} + 5 \times$

$\overline{b^2 + ba + a^2} + 10 \times \overline{b + a} + 10$, or $b^7 + ab^6 + a^2b^5$

$+ a^3b^4 + a^4b^3 + a^5b^2 + a^6b + a^7 + 4b^6 + 4ab^5 + 4a^2b^4$

$+ 4a^3b^3 + 4a^4b^2 + 4a^5b + 4a^6 + 5b^3 + 5ab^2 + 5a^2b$

$+ 5a^3 = b^5 + ab^4 + a^2b3 + a3b^2 + a^4b + a^5 + 10b4$

$+ 10ab^3 + 10a^2b^2 + 10a3b + 10a^4 + 5b^2 + 5ab + 5a^2$

$+ 10b + 10a \div 10$, or

$$\left\{\begin{array}{l} b^7 + ab^6 + a^2b^5 + a^3b^4 + a^4b^3 + a^5b^2 + a^6b + a^7 \\ \quad + 4b^6 + 4ab^5 + 4a^2b^4 + 4a^3b^3 + 4a^4b^2 + 4a^5b + 4a^6 \\ \qquad\qquad + 5b^3 + 5ab^2 + 5a^2b + 5a^3 \end{array}\right\}$$

$$= \left\{\begin{array}{l} b^5 + ab^4 + a^2b^3 + a^3b^2 + a^4b + a^5 \\ \quad + 10b^4 + 10ab^3 + 10a^2b^2 + 10a^3b + 10a^4 \\ \qquad\qquad + 5b^2 + 5ab + 5a^2 \\ \qquad\qquad\qquad + 10b + 10a \\ \qquad\qquad\qquad\qquad + 10, \end{array}\right.$$

and (ſubtracting $a^5 + 10a^4 + 5a^2 + 10a + 10$ from both ſides,)

$b^7 +$

$$\left\{\begin{array}{l} b^7+ab^6+a^2b^5+a^3b^4+a^4b^3+a^5b^2+a^6b+a^7 \\ \quad +4b^6+4ab^5+4a^2b^4+4a^3b^3+4a^4b^2+4a^5b+4a^6 \\ \qquad\qquad +5b^5+5ab^4+5a^2b^3+5a^3b^2+5a^4b+5a^5 \\ \qquad\qquad\qquad\qquad -a^5 \\ \qquad\qquad\qquad\qquad -10a^4 \\ \qquad\qquad\qquad\qquad -5a^3 \\ \qquad\qquad\qquad\qquad -10a^2 \\ \qquad\qquad\qquad\qquad -10a \end{array}\right.$$

$$= \left\{\begin{array}{l} b^5+ab^4+a^2b^3+a^3b^2+a^4b \\ \quad +10b^4+10ab^3+10a^2b^2+10a^3b \\ \qquad\qquad +5b^3+5ab^2 \\ \qquad\qquad\qquad +10b, \end{array}\right.$$

and (ſubtracting from both ſides of this equation all the terms on the left-hand ſide of it that involve the powers of b, and placing all the terms that involve the powers of b on the left-hand ſide of the new equation,)

$$\left\{\begin{array}{l} \quad 10b+5b^2+10ab^3+10b^4+b^5-4b^6-b^7 \\ +\ 5ab+10a^2b^2+a^2b^3+ab^4-4ab^5-ab^6 \\ +\ 10a^3b+a^3b^2-5b^3-4a^2b^4-a^2b^5 \\ +\ a^4b-5ab^2-4a^3b^3-a^3b^4 \\ -\ 5a^2b-4a^4b^2-a^4b^3 \\ -\ 4a^5b-a^5b^2 \\ -\ a^6b \end{array}\right.$$

$= a^7+4a^6-a^5-10a^4+5a^3-5a^2-10a-10,$

or (if we range the terms in the ſeveral vertical columns on the left-hand ſide of this equation according to the powers of a,)

$$\left\{\begin{array}{l} \quad 10b + 5b^2 - 5b^3 + 10b^4 + b^5 - 4b^6 - b^7 \\ + 5ab - 5ab^2 + 10ab^3 + ab^4 - 4ab^5 - ab^6 \\ - 5a^2b + 10a^2b^2 + a^2b^3 - 4a^2b^4 - a^2b^5 \\ + 10a^3b + a^3b^2 - 4a^3b^3 - a^3b^4 \\ + a^4b - 4a^4b^2 - a^4b^3 \\ - 4a^5b - a^5b^2 \\ - a^6b \end{array}\right\}$$

$$= a^7 + 4a^6 - a^5 - 10a^4 + 5a^3 - 5a^2 - 10a - 10.$$

But the octinomial quantity $a^7 + 4a^6 - a^5 - 10a^4 + 5a^3 - 5a^2 - 10a - 10$ is $=$

$$\frac{a^8 + 4a^7 - a^6 - 10a^5 + 5a^4 - 5a^3 - 10aa - 10a}{a} =$$

$$\frac{5}{a} = \frac{5}{1.618} = 3.090.$$

Therefore the complicated quantity

$$\left\{\begin{array}{l} \quad 10b + 5b^2 - 5b^3 + 10b^4 + b^5 - 4b^6 - b^7 \\ + 5ab - 5ab^2 + 10ab^3 + ab^4 - 4ab^5 - ab^6 \\ - 5a^2b + 10a^2b^2 + a^2b^3 - 4a^2b^4 - a^2b^5 \\ + 10a^3b + a^3b^2 - 4a^3b^3 - a^3b^4 \\ + a^4b - 4a^4b^2 - a^4b^3 \\ - 4a^5b - a^5b^2 \\ - a^6b \end{array}\right\}$$

will be $= 3.090$.

Art. 52. Now let 1.618 be substituted in the terms of this equation instead of a.

And

And we shall have aa ($= \overline{1.618}^2$) $= 2.617,924$, and $a^3 = 4.235,801$, and $a^4 = 6.853,526$, and $a^5 = 11.089,005$, and $a^6 = 17.942,010$, and $a^7 = 29.030,172$, and $a^8 = 46.970,818$, and consequently $5a$ ($= 5 \times 1.618$) $= 8.090$, and $5a^2$ ($= 5 \times 2.617,924$) $= 13.089,620$, and $10a^3$ ($= 10 \times 4.235,801$) $= 42.358,010$, and $4a^5$ ($= 4 \times 11.089,005$) $= 44.356,020$, and $10b + 5ab - 5a^2b + 10a^3b + a^4b - 4a^5b - a^6b$ ($= 10b + 8.090 \times b - 13.089,620 \times b + 42.358,010 \times b + 6.853,526 \times b - 44.356,020 \times b - 17.942,010 \times b$) $= 67.301,536 \times b - 75.387,650 \times b$; and $10a^2$ ($= 10 \times 2.617,924$) $= 26.179,240$, and $4a^4$ ($= 4 \times 6.853,526$) $= 27.414,104$, and $5b^2 - 5ab^2 + 10a^2b^2 + a^3b^2 - 4a^4b^2 - a^5b^2$ ($= 5b^2 - 8.090 \times b^2 + 26.179,240 \times b^2 + 4.235,801 \times b^2 - 27.414,104 \times b^2 - 11.089,005 \times b^2$) $= 30.415,041 \times b^2 - 46.593,109 \times b^2$; and $10a$ ($= 10 \times 1.618$) $= 16.18$, and $4a^3$ ($= 4 \times 4.235,801$) $= 16.943,204$, and $-5b^3 + 10ab^3 + a^2b^3 - 4a^3b^3 - a^4b^3$ ($= -5b^3 + 16.18 \times b^3 + 2.617,924 \times b^3 - 16.943,204 \times b^3 - 6.853,526 \times b^3$) $= -28.796,730 \times b^3 + 18.797,924 \times b^3$; and $4a^2$ ($= 4 \times 2.617,924$) $= 10.471,696$, and $10b^4 + ab^4 - 4a^2b^4 - a^3b^4$ ($= 10b^4 + 1.618 \times b^4 - 10.471,696 \times b^4 - 4.235,801 \times b^4$) $= 11.618,000 \times b^4 - 14.707,497 \times b^4$; and $4a$ ($= 4 \times 1.618$) $= 6.472$, and $b^5 - 4ab^5 - a^2b^5$ ($= b^5 - 6.472,000 \times b^5 - 2.617,924 \times b^5$) $= b^5 - 9.089,924 \times b^5$; and $-4b^6 - ab^6$ ($= -4.000,000 \times b^6 - 1.618,000 \times b^6$) $= -5.618,000 \times b^6$. Therefore the left-hand side of the foregoing equation will be equal to the following compound quantity, to wit,

67.301,

67.301,536 × b — 75.387,650 × b + 30.415,041 × b^2 — 46.593,109 × b^2 — 28.796,730 × b^3 + 18.797,924 × b^3 + 11.618,000 × b^4 — 14.707,497 × b^4 + 1.000,000 × b^5 — 9.089.924 × b^5 — 5.618,000 × b^6 — b^7. Therefore this laſt compound quantity muſt be equal to 3.090; or, in other words, the five quantities 67.301,536 × b + 30.415,041 × b^2 + 18.797,924 × b^3 + 11.618,000 × b^4 + 1.000,000 × b^5 muſt be greater than the ſeven quantities 75.387,650 × b + 46.593,109 × b^2 + 28.796,730 × b^3 + 14.707,497 × b^4 + 9.089,924 × b^5 + 5.618,000 × b^6 + b^7, and the exceſs muſt be equal to 3.090. But the ſaid five quantities are, reſpectively, leſs than the five firſt quantities of the ſaid ſeven quantities, and therefore, *à fortiori*, muſt be leſs than all the ſaid ſeven quantities, and therefore the ſaid ſeven quantities cannot be ſubtracted from the ſaid five quantities, as they are ſuppoſed to be in the ſaid laſt equation. And conſequently the ſaid laſt equation is impoſſible. Therefore the ſuppoſition from which the ſaid equation is derived, to wit, the ſuppoſition that the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$ has a ſecond real and affirmative root b, greater than a, or 1.618, muſt be a falſe ſuppoſition. And conſequently we may, at laſt, conclude that 1.618, or 1.618,037, is the only root of the propoſed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

Q. E. I.

Art. 53. This root, 1.618, is pretty near the truth. For, if we ſuppoſe x to be = 1.618, we ſhall have

x^8

$x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$ (= 46970,818 + 4 × 29.030,172 — 17.942,010 — 10 × 11.089,005 + 5 × 6.853,526 — 5 × 4.235,801 — 10 × 2.617,924 — 10 × 1.618) = 46.970,818 + 116.120,688 — 17.942,010 — 110.890,050 + 34.267,630 — 21.179,005 — 26.179,240 — 16.180,000 = 197.359,136 — 192.370,305 = 4.988,831; which is less than 5, or the absolute term of the proposed equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, by only the small difference 0.011,169, which is less than the 447th part of the absolute term 5. Therefore 1.618 must be somewhat less than, but very nearly equal to, the true value of x in the said equation. Q. E. D.

A SCHOLIUM.

Art. 54. We have now gone through the resolution of three equations consisting of terms marked with the signs — and + alternately, (and which therefore are capable of having more than one real and affirmative root, if the co-efficients of the powers of x and the absolute terms of the equations are of certain relative magnitudes, with respect to each other, that are proper

for that purpoſe,) to wit, the cubick equation $x^3 - 39x^2 + 479x = 1881$, and the equation $x^5 - 7x^4 + 20x^3 - 155xx = 10{,}000$, and the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x = 5$, by means of the proceſſes deſcribed above in Art. 9, 10, 11, 12, and 13, together with Mr. Raphſon's method of approximation, after a tolerably near value of the leaſt root of each of thoſe equations, ſomewhat leſs than the true value of it, had been firſt obtained by thoſe proceſſes. And theſe proceſſes have been ſet forth ſo fully and diſtinctly that nothing more is neceſſary, as I conceive, to be added, in order to explain and illuſtrate them. And I have likewiſe ſhewn, at the end of each of theſe examples, a method of reducing, when ſuch reduction is poſſible, and of attempting, in other caſes, to reduce, the equation (after it's leaſt root, or it's only root, a, has been obtained, to a ſufficient degree of exactneſs, by means of the ſaid proceſſes and of Mr. Raphſon's method of approximation,) to an equation one degree lower than the firſt equation, by putting b for one of it's other roots, which are greater than a, (if it has any ſuch other roots,) and dividing the terms of the equation reſulting from that ſuppoſition by the reſidual quantity $b - a$. And by this reduction, or attempt at a reduction, of the ſaid equation to an equation one degree lower, we can uſually diſcover whether the original equation has, or has not, any other root than a. For, if the equation obtained by this diviſion is a poſſible equation (as was the caſe in the firſt of the three foregoing examples, where the equation ſo obtained was the quadratick equation $30x - xx = 209$, which is a poſſible equation, and has two roots,) the original equation, from which it was derived, will have more roots

than a, to wit, the root, or roots, of the ſaid ſecond equation, which was derived from it by means of the ſaid diviſion: but, if the ſaid ſecond equation appears to be impoſſible (as was the caſe in the ſecond and third of the three foregoing examples,) the original equation, from which it was derived by means of the ſaid diviſion, can have no other root than the root a, which has been already found. And in the former caſe, or when the ſaid ſecond equation is found to be a poſſible one, it's leaſt root may be found by a repetition of the proceſſes above-deſcribed, together with Mr. Raphſon's method of approximation, in the ſame manner as the leaſt root of the original equation had been obtained before. And then the next greater root of the ſaid ſecond equation, and the next greater than that, and ſo on, may be found in the ſame manner, till all it's roots are known; which are likewiſe roots of the original equation from which the ſaid ſecond equation was derived. And thus at laſt all the roots of the ſaid original equation will be obtained: which is all that can be deſired on the ſubject.

Art. 55. But I am apprehenſive that it may be thought that the number of proceſſes to be uſed in this method is too great, ſo as to make the inveſtigation of the value of a, or the leaſt root of the propoſed equation, a tedious and troubleſome buſineſs. That it is ſo, in ſome degree, I readily confeſs: but I apprehend that this difficulty and tediouſneſs are inherent in the ſubject itſelf, and inſeparable from it, and that the leaſt root, or any other root, of an equation of the eighth degree, that has all it's terms compleat (as is the caſe with the equation

equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$,) cannot be found, by any method hitherto diſcovered, with leſs trouble than was taken in the foregoing inveſtigation of the root of the ſaid equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$. And it muſt be obſerved that the proceſſes by which the near value 1.6 (which was made the baſis of the further approximation to the root of the ſaid equation by Mr. Raphſon's method of approximation,) was obtained, though they were pretty many in number, were very ſimple and eaſy arithmetical operations. For they were only ſubſtitutions of the ſeveral ſhort numbers 1, 0.7, 0.9, 1.2, 1.3, 1.4, and 1.6, (conſiſting of only one and two figures,) inſtead of y, and a, and c, and x, in the compound quantities $y^8 + 4y^7 + 5y^4$, and $5 + a^6 + 10a^5 + 5a^3 + 10a^2 + 10a$, and $5 + c^6 + 10c^5 + 5c^3 + 10c^2 + 10c$, and $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x$. The remaining part of the inveſtigation of the value of a, or of the leaſt root of the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, after we have obtained 1.6 for a near value of it ſomewhat leſs than the truth, is performed by Mr. Raphſon's method of approximation, which is ſubſtantially the ſame with Sir Iſaac Newton's, and is allowed on all hands to be the ſhorteſt and beſt method that can be taken for the purpoſe. I therefore conceive that the only anſwer that can be given to this complaint of the tedioufneſs of the foregoing method of obtaining the leaſt root of an equation of a high degree, is that which is ſaid to have been given by Euclid, the author of the Elements of

Geometry, to Ptolemy Philadelphus, King of Egypt, when he made a like complaint concerning the difficulty of that method of arriving at the knowledge of Geometry, to wit, "*That there is no royal highway to the knowledge of Geometry, and much leſs to that of the method of reſolving an equation of the eighth degree, or of any higher order.*"

End of the Scholium begun in Art. 54.

Art. 56. In the foregoing three equations which we have reſolved in the method deſcribed above in art. 10, 11, 12, 13, and 14, in order to illuſtrate the ſaid method, to wit, the cubick equation $x^3 - 39x^2 + 479x = 1881$, and the equation $x^5 - 7x^4 + 20x^3 - 155x^2 = 10{,}000$, and the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$, the higheſt power of x is marked with the ſign $+$, or the ſum of the higheſt power of x and the terms that are added to it is greater than the ſum of the other terms of the equation involving the other powers of x, and the latter ſum is ſubtracted from the former ſum, and the terms that compoſe it are conſequently marked with the ſign $-$. But the method deſcribed in art. 10, 11, 12, 13, and 14, is equally uſeful in inveſtigating the firſt near value of the leaſt root of an equation that has ſeveral roots, when the higheſt power of x is marked with the ſign $-$; of which it may be proper, before I conclude this diſcourſe, to give an example. I ſhall therefore now proceed to apply this method to the reſolution of an equation of this kind; and the equation I ſhall chuſe for this purpoſe ſhall be the biquadratick equation $14{,}937x - 1998x^2 + 80x^3$

$- x^4 = 5000$, which we have already reſolved in the former part of this volume, and which reſulted from Dr. Wallis's Solution of Colonel Titus's Arithmetical Problem above-mentioned. This equation has been already examined at great length in the former part of this volume in the Appendix to Dr. Halley's Tract; and it has been ſhewn to have four different real and affirmative roots, to wit, the numbers 0.350,987, &c, 12.756,441, 794,480,744,022, &c, 32.060,290,8, &c, and 34.832, 280, &c, of which the ſecond, or leaſt but one, to wit, 12.756,441,794,480,744,022, &c, was inveſtigated in pages 65, 66, 67, &c - - - 96, and the other three were inveſtigated afterwards in pages 144, 145, 146, &c - - - 175. And in page 161 it is aſſerted that the firſt, or leaſt, of theſe roots, to wit, 0.359,987, &c, would, if computed to a greater degree of exactneſs, be found to be equal to 0 350,987,045,866,14. This leaſt root I now propoſe to inveſtigate by the method above-deſcribed, ſo as to find a firſt near value of it that ſhall be ſufficiently near to it's true value to be made uſe of as a convenient ground, or baſis, of a further approximation to the ſaid true value in the manner recommended by Mr. Raphſon; and I ſhall afterwards, by means of Mr. Raphſon's method, continue the inveſtigation of it to twelve or thirteen places of decimal figures, but ſhall ſay nothing more of the other three roots of this equation, which have been ſufficiently conſidered in the former part of this volume.

Art. 57. This leaſt root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ is the more worthy of

our attention and of a more accurate inveſtigation than has been already beſtowed on it, becauſe it is of uſe in the ſolution of Colonel Titus's Problem above-mentioned, as well as the ſecond root 12.756,441,794,480,744,023, &c which Dr. Halley has inveſtigated: for it enables us (as we have ſeen above in pages 234, 235, and 236,) to find the ſecond ſet of numbers, or of values of a, b, and c, in Colonel Titus's Problem, that will anſwer the conditions of that problem, to wit, the numbers 0.418,919,470, &c, 3.912,226,866, &c, and 4.044,884,670, &c. This leaſt root of the ſaid equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ I ſhall therefore now proceed to inveſtigate in the method deſcribed above in art. 10, 11, 12, 13, and 14.

An Inveſtigation of the Leaſt Root of the Biquadratick Equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ *by Means of the Proceſſes deſcribed in Art.* 10, 11, 12, 13, *and* 14, *and Mr. Raphſon's Method of Approximation.*

Art. 58. In the firſt place we will ſeek the root of the curtailed equation $14{,}937x + 80x^3 = 5000$, or $14{,}937y + 80y^3 = 5000$, which has but one root. And this one root we will ſeek by conjectures and trials made in the manner directed by Stevinus and Kerſey. This may be done as follows:

Let

Let us firſt ſuppoſe y to be $= 1$, and try the effect of this ſuppoſition.

Now, if y is $= 1$, we ſhall have y^3 alſo $= 1$, and and $80y^3$ $(= 80 \times 1) = 80$. Therefore $14{,}937y + 80y^3$ will in this caſe be $(= 14{,}937 + 80) = 15{,}017$; which is much greater than 5000, or the abſolute term of the equation $14{,}937y + 80y^3 = 5000$. Therefore 1 muſt be much greater than the true value of y in this equation, or the ſaid true value of y muſt be much leſs than 1.

Secondly, ſince y is leſs than 1, yy will be leſs than y, and y^3 will be leſs than yy, and, *à fortiori*, leſs than y. Therefore $80y^3$ will be much leſs than $14{,}937y$, and conſequently $14{,}937y$ alone will be nearly equal to $14{,}937y + 80y^3$, and therefore to (it's equal,) 5000. Therefore y will be nearly equal to $\frac{5000}{14{,}937}$, or to 0.33. Therefore, (by the Lemma demonſtrated in art. 10,) 0.33 (which is nearly equal to the ſingle root of the equation $14{,}937y + 80y^3 = 5000$,) muſt be leſs than the leaſt root of the original equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and may therefore, if it is near enough to the true value of the ſaid leaſt root, be conveniently made uſe of as the ground, or baſis, of a further approximation to the true value of the ſaid leaſt root by putting $x = 0.33 + z$, and ſubſtituting $0.33 + z$ inſtead of x in the terms of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and reſolving the tranſformed equation reſulting from ſuch ſubſtitution as if it

were a mere ſimple equation, according to the directions of Mr. Raphſon.

Art. 59. We muſt therefore in the next place endeavour to diſcover whether 0.33 is near enough to the true value of x, or the leaſt root of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$ to be fit to be made uſe of in this manner. And for this purpoſe we muſt ſubſtitute it inſtead of x in the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$, and compare the value of the ſaid quadrinomial quantity reſulting from ſuch ſubſtitution with the abſolute term 5000, to which the ſaid quadrinomial quantity is equal in the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$.

Now, if x is $= 0.33$, we ſhall have x^2 $(= \overline{0.33}^2)$ $= 0.1089$, and x^3 $(= \overline{0.33}^3)$ $= 0.035{,}937$, and x^4 $(= \overline{0.33}^4)$ $= 0.011{,}859{,}21$, and conſequently $14{,}937x$ $(= 14{,}937 \times 0.33)$ $= 4929.21$, and $1998x^2$ $(= 1998 \times 0.1089)$ $= 217.5822$, and $80x^3$ $(= 80 \times 0.035{,}937)$ $= 2.874{,}960$. Therefore the whole compound quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be $(= 4929.21 - 217.5822 + 2.874{,}960 - 0.011{,}859{,}21 = 4932.084{,}960 - 217.594{,}059{,}21) = 4714.490{,}900{,}79$; which is conſiderably leſs than 5000, or the abſolute term of the propoſed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.33 will be conſiderably leſs than the true value of x, or the leaſt root of that equation: and therefore it will be expedient to form another conjecture concerning the ſaid true value of x, that will be rather nearer to it. And, on account of the magnitude of the difference of the foregoing reſult

4714.

4714.490,900,79 from the abſolute term 5000, it ſeems reaſonable to conjecture that the ſaid true value will be very nearly equal to 0.35. We will therefore ſuppoſe x to be equal to 0.35, and will ſubſtitute 0.35 inſtead of x in the terms of the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than the abſolute term 5000, and conſequently whether 0.35 will be greater, or leſs, than the true value of x, or the leaſt root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, and likewiſe whether the ſaid reſult will approach near enough to the ſaid abſolute term to make it expedient to make uſe of 0.35 as the ground, or baſis, of a further approximation to the true value of the ſaid leaſt root in the manner recommended by Mr. Raphſon.

Art. 60. Now, if x is $= 0.35$, we ſhall have $x^2 (= \overline{0.35}^2) = 0.1225$, and $x^3 (= \overline{0.35}^3) = 0.042,875$, and $x^4 (= \overline{0.35}^4) = 0.015,006,25$, and conſequently $14,937x$ $(= 14,937 \times 0.35) = 5227.95$, and $1998x^2$ $(= 1998 \times 0.1225) = 244.7550$, and $80x^3$ $(= 80 \times 0.042,875) = 3.430,000$, and conſequently $14,937x - 1998x^2 + 80x^3 - x^4$ $(= 5227.95 - 244.7550 + 3\,430,000 - 0.015,006,25) = 5231.380,000 - 244.770,006,25) = 4986.609,993,75$; which is ſomewhat leſs than 5000, or the abſolute term of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$; but the difference between them is but ſmall. Therefore 0.35 will be ſomewhat leſs than the true value of x, or the ſaid equation; but the difference between them will be but ſmall. Q. E. I.

Art. 61.

Art. 61. We will therefore now take 0.35 for the baſis of a further approximation towards the true value of x, or the leaſt root of the propoſed equation, in the manner recommended by Mr. Raphſon, and ſuppoſe x to be $= 0.35 + z$, and ſubſtitute the ſaid binomial quantity inſtead of x in the terms of the ſaid equation. This may be done in the following manner:

Since x is $= 0.35 + z$, we ſhall have x^2 $(= \overline{0.35 + z}^2 = \overline{0.35}^2 + 2 \times 0.35 \times z + \&c = \overline{0.35}^2 + 0.70 \times z + \&c) = 0.1225 + 0.70 \times z + \&c$,

and x^3 $(= \overline{0.35 + z}^3 = \overline{0.35}^3 + 3 \times \overline{0.35}^2 \times z + \&c = \overline{0.35}^3 + 3 \times 0.1225 \times z + \&c = \overline{0.35}^3 + 0.3675 \times z + \&c)$ $= 0.042,875 + 0.3675 \times z + \&c$,

and x^4 $(= \overline{0.35 + z}^4 = \overline{0.35}^4 + 4 \times \overline{0.35}^3 \times z + \&c = \overline{0.35}^4 + 4 \times 0.042,875 \times z + \&c = \overline{0.35}^4 + 0.171,500 \times z + \&c) = 0.015,006,25, + 0.171,500 \times z + \&c$.

Therefore $14,937x$ will be $(= 14,937 \times \overline{0.35 + z} = 14,937 \times 0.35 + 14,937 \times z) = 5227.95 + 14,937z$; and $1998x^2$ will be $(= 1998 \times \overline{0.1225 + 0.70 \times z + \&c} = 1998 \times 0.1225 + 1998 \times 0.70 \times z + \&c) = 244.7550 + 1,398.60 \times z$; and

and $80x^3$ will be $(= 80 \times \overline{0.042,875 + 0.3675 \times z} + \&c$ $= 80 \times 0.042,875 + 80 \times 0.3675 \times z + \&c) =$ $3.430,000 + 29.4000 \times z + \&c$; and consequently the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{rrrr} 5227.95 & + & 14,937 \times z & \\ - \ 244.7550 & - & 1398.60 \times z - \&c & \\ + \ 3.430,000 & + & 29.4000 \times z + \&c & \\ - \ 0.015,006,25 & - & 0.171,500 \times z - \&c & \end{array}\right\}$$

$$= \left\{\begin{array}{rrrr} 5231.380,000,00 & + & 14,966.400,000 \times z + \&c & \\ - \ 244.770,006,25 & - & 1,398.771,500 \times z - \&c & \end{array}\right\}$$

$$= 4986.609,993,75 + 13,567.628,590 \times z \ \&c.$$

But the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the quantity $4986.609,993,75 + 13,567.628,500 \times z$ &c will alſo be $= 5000$. And conſequently $13,567.628,500 \times z$ will be $(= 5000.000,000,00 - 4986.609,993,75) = 13.390,006,25$, and z will be $\left(= \frac{13.390,006,25}{13,567.628,500}\right) = 0.000,986,9$. Therefore x, or $0.35 + z$, will be $(= 0.35 + 0.000,986,9) = 0.350,986,9$, or, very nearly, $0.350,987$; that is, $0.350,987$ will be another near value of x, or the leaſt root of the propoſed equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$, that will be much nearer than 0.35, or it's laſt near value, to it's true value. Q. E. I.

Art. 62.

Art. 62. Now, in order to discover how near this last number, 0.350,987, will approach to the true value of x, or the least root of the proposed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and likewise to discover whether it will be greater, or less, than the said true value, let 0.350,987 be substituted instead of x in the terms of the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$. This may be done as follows:

If x is $= 0.350{,}987$, we shall have $x^2 = 0.123,191,874,169$, and $x^3 = 0.043,238,746,338,954,803$, and $x^4 = 0.015,176,237,861,270,729,440,561$, and consequently $14{,}937x$ $(= 14{,}937 \times 0.350{,}987) = 5242.692{,}819$, and $1998x^2$ $(= 1998 \times 0.123,191,874,169) = 246.137,364,589,662$, and $80x^3$ $(= 80 \times 0.043,238,746,338,954,803) = 3.459,099,707,116,384,240$. Therefore the whole quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ will be $(= 5242.692{,}819, - 246.137,364,589,662 + 3.459,099,707,116,384,240 - 0.015,176,237,861,270,729,440,561 = 5246.151,918,707,116,384,240 - 246.152,540,827,523,270,729,440,561) = 4999.999,377,879,593,113,510,559,439$; which is less than 5000, or the absolute term of the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$. Therefore 0.350,987 is less than the true value of x, or the least root of that equation. Q. E. I.

Art. 63. We will therefore now put $x = 0.350{,}987 + z$, and substitute this binomial quantity instead of x in the proposed equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, and resolve the transformed equation thence arising

ariſing in the manner preſcribed by Mr. Raphſon, in order to obtain a ſtill nearer value of x, or the leaſt root of the ſaid equation, than 0.350,987. This may be done in the manner following:

Since x is $= 0.350,987 + z$, we ſhall have x^2 $(= \overline{0.350,987 + z}^2 = \overline{0.350,987}^2 + 2 \times 0.350,987 \times z + \&c = \overline{0.350,987}^2 + 0.701,974 \times z + \&c)$ $= 0.123,191,874,169 + 0.701,974 \times z + \&c$,

and x^3 $(= \overline{0.350,987 + z}^3 = \overline{0.350,987}^3 + 3 \times \overline{0.350,987}^2 \times z + \&c = \overline{0.350,987}^3 + 3 \times 0.123,191,874,169 \times z + \&c = \overline{0.350,987}^3 + 0.369,575,622,507 \times z + \&c)$ $= 0.043,238,746,338,954,803 + 0.369,575,622,507 \times z + \&c$,

and x^4 $(= \overline{0.350,987 + z}^4 = \overline{0.350,987}^4 + 4 \times \overline{0.350,987}^3 \times z + \&c = \overline{0.350,987}^4 + 4 \times 0.043,238,746,338,954,803 \times z + \&c = \overline{0.350,987}^4 + 0.172,954,985,355,819,212 \times z + \&c)$ $= 0.015,176,237,861,270,729,440,561 + 0.172,954,985,355,819,212 \times z + \&c$.

Therefore $14,937x$ will be $(= 14,937 \times \overline{0.350,987+z}$ $= 14,937 \times 0.350,987 + 14,937 \times z)$ $= 5242.692,819 + 14,937z$;

and

and $1998x^2$ will be $(= 1998 \times$
$0.123,191,874,169 + 0.701,974 \times z + \&c$
$= 1998 \times 0.123,191,874,169 + 1998 \times$
$0.701,974 \times z + \&c) = 246.137,364,$
$589,662 + 1402.544,052 \times z + \&c;$

and $80x^3$ will be $(= 80 \times 0.043,238,746,338,954,803$
$+ 80 \times 0.369,575,622,507 \times z + \&c)$
$= 3.459,099,707,116,384,240 + 29.566,$
$049,800,560 \times z + \&c;$

and consequently the whole quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ will be equal to the following compound quantity, to wit,

$$\left\{\begin{array}{llll} 5242.692,819 & & + 14,937 \times z & \\ -246.137,364,589,662 & & - 1402.544,052 \times z & - \&c \\ + 3.459,099,707,116,384,240 & + & 29.566,049,800,560 \times & \\ - 0.015,176,237,861,270,729,440,561 & & z + \&c & \\ & - 0.172,954,985,355,819,212 \times z & - \&c & \end{array}\right\}$$

$$= \left\{\begin{array}{l} 5246.151,918,707,116,384,240 \\ \qquad + 14,966.516,049,800,560, \times z + \&c \\ -246.152,540,827,523,270,729,440,561 \\ \qquad - 1402.717,006,985,355,819,212 \times z - \&c \end{array}\right\}$$

$= 4999.999,377,879,593,113,510,559,439 - 13,563.$
$.849,042,815,204,180,788 \times z \ \&c.$

But the quadrinomial quantity $14,937x - 1998x^2 + 80x^3 - x^4$ is $= 5000.$

Therefore

Therefore the quantity 4999.999,377,879,593,113,510,559,439 + 13,563.849,042,815,204,180,788 × z &c will alſo be = 5000; and conſequently the quantity 13,563.849,042,815,204,180,788 × z &c will be =

$$\left\{\begin{array}{r} 5000.000,000,000,000,000,000,000,000 \\ -\ 4999.999,377,879,593,113,510,559,439 \end{array}\right\}$$
$$= \quad 0.000,622,120,406,886,489,440,561,$$

and z will be $= \frac{0.000,622,120,406,886,489,440,561,}{13,563.849,042,815,204,180,788} =$ 0.000,000,045,866,06 &c. Therefore x, or 0.350,987 + z, will be = 0.350,987 + 0.000,000,045,866,06 &c = 0.350,987,045,866,06 &c. Q. E. I.

Art. 64. This value of x differs from the number 0.350,987,045,866,14 (to which I had aſſerted in page 161 that the leaſt root of the equation $14,937x - 1998x^2 + 80x^3 - x^4 = 5000$ would be found to be equal,) in the 13th place of decimal fractions; ſo that I muſt have made a ſlip in one of the two calculations by which I obtained theſe numbers in that place of figures. But, as this difference of the two numbers occurs only in the 13th figure, we may ſafely conclude that the firſt twelve figures 0.350,987,045,866, of the number 0.350,987,045,866,06, that has juſt now been obtained by the foregoing inveſtigation, which are the ſame in both calculations, will be exact.

End of the General Method of inveſtigating the Two or Three Firſt Figures of the Leaſt Root of an Equation that has more than One Real and Affirmative Root.

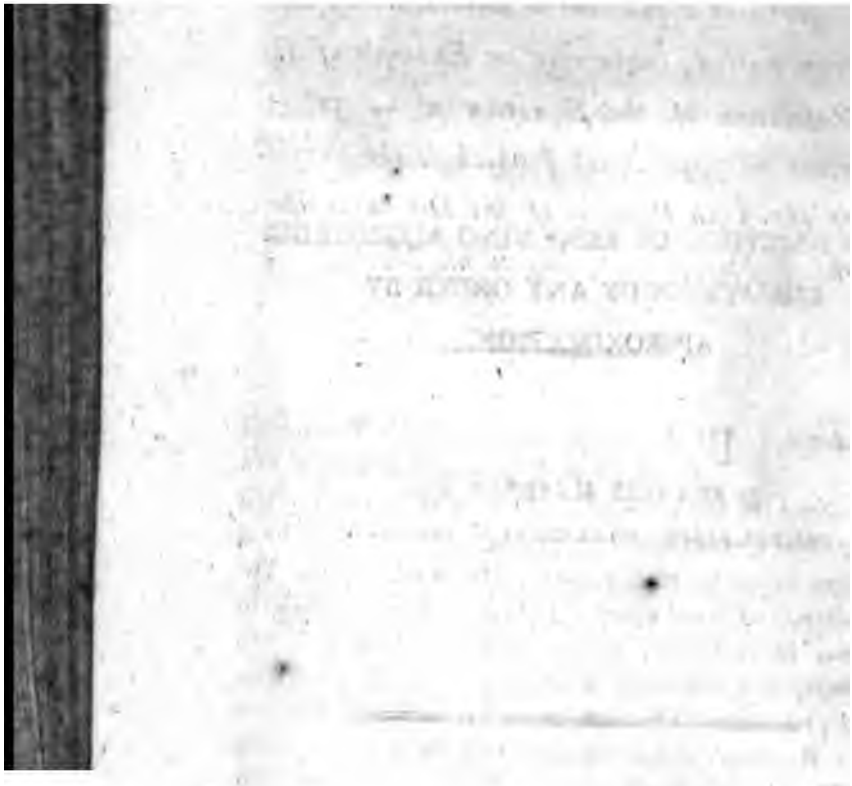
A SPECIMEN

OF

VIETA's METHOD OF RESOLVING ALGEBRAICK EQUATIONS OF ANY ORDER BY APPROXIMATION.

By FRANCIS MASERES, Esq.

CURSITOR BARON OF THE COURT OF EXCHEQUER.



A Specimen *of* Vieta's *Method of reſolving Algebräick Equations of any Order by Approximation; containing an Example of the Reſolution of the Equation* $x^5 - 5x^3 + 500x = 7{,}905{,}504$, *(which is the Subject of the* 15*th Problem of his Diſcourſe upon this Subject,) according to his Method.*

Article 1. IN the foregoing part of this Volume of Tracts I have endeavoured to explain and illuſtrate the three ſeveral methods of reſolving Affected Algebräick equations by approximation that have been given us by Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley, and have compared them with each other, in order to enable my readers to judge of their ſeveral merits, and determine to which of them they will give the preference. This compariſon has been made between Mr. Raphſon's and Dr. Halley's methods in the Appendix to Dr. Halley's Tract on this ſubject, which forms the Second Tract in this collection, and extends from page 25 to page 183. In this Appendix I have reſolved each of the three equations which Dr. Halley has brought as

examples of his method of reſolution, to wit, the cubick equation $x^3 - 17x^2 + 54x = 350$, and the biquadratick equations $x^4 - 3x^2 + 75x = 10{,}000$ and $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, in two different manners, to wit, firſt, in the manner preſcribed by Dr. Halley, and, ſecondly, in the manner preſcribed by Mr. Raphſon, that the advantages and diſadvantages of each method may be the more apparent: and I have ſtated my reaſons for thinking that, in moſt caſes, Mr. Raphſon's method deſerves to be preferred to Dr. Halley's. And I have compared Sir Iſaac Newton's method of reſolution with that of Mr. Raphſon in the Tract intitled "*Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all Degrees by Approximation*," which extends from page 279 to page 323; in which Tract I have reſolved the only equation that has been brought by Sir Iſaac Newton as an example of his method, to wit, the cubick equation $x^3 - 2x = 5$, in two different manners, to wit, firſt, in the manner preſcribed by Sir Iſaac Newton, and, ſecondly, in the manner preſcribed by Mr. Raphſon; and have compared the methods with each other in the ſeveral proceſſes of them, ſtep by ſtep, and mentioned my reaſons for concluding that Mr. Raphſon's method, though it differs but little from Sir Iſaac Newton's, yet deſerves, in ſome degree, to be preferred to it. And, as in all theſe three methods, of Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley, it is preſuppoſed that the calculator is already able to find the value of a, (or the firſt near value of x, or the root ſought,) to a tolerable degree of exactneſs, (or to within one hundredth part, or, at leaſt, to within one tenth part, of the true value of

of x,) either by conjecture and trial, or otherwiſe, which firſt near value, a, is to be made the ground, or baſis, of a further approach to the true value of x by their ſeveral methods of approximation;—and, as neither of thoſe eminent writers has given his readers any directions as to the manner of finding ſuch firſt near value;—I have endeavoured to ſupply this defect by re-printing the 10th Chapter of Mr. John Kerſey's excellent Treatiſe on Algebra, in which that clear and eaſy writer has ſet-forth and illuſtrated by examples the general method given us by Simon Stevinus, (the Flemiſh Mathematician of the former part of the laſt century,) for that purpoſe: and to this method I have ſubjoined a little improvement of my own, which will be of uſe in ſome particular caſes, which might otherwiſe create ſome doubt and perplexity: I mean thoſe caſes in which ſome of the terms which involve the powers of x in the equation are marked with the ſign $+$, and others of them are marked with the ſign $-$, and in which it is conſequently often probable that the equation that is to be reſolved may have more than one real and affirmative root; in all which caſes the improvement here alluded-to will enable us to find a quantity that will always be leſs than the leaſt of the ſeveral different roots of the equation, if it has more than one ſuch root, or than the only root of it, if, notwithſtanding the appearances to the contrary, it ſhould have only one ſuch root; and from this quantity (which is always leſs than the true value of the ſaid leaſt, or only, root,) we may make gradual approaches, by Stevinus's method, or otherwiſe, to the ſaid true value, till we are come near enough to it to make it expedient

to proceed afterwards to greater degrees of exactneſs by Mr. Raphſon's method of approximation. And in this laſt Tract I have likewiſe ſhewn how, when the leaſt, or only, root of the propoſed equation has been ſo diſcovered, we may, if the equation has no other root beſides that which has been ſo diſcovered, determine with certainty that it has no other root, and, if it has one or more other roots, reduce the equation to another equation one degree lower than the former, which will involve the ſaid other root or roots of the firſt equation; after which we may, by performing the like proceſſes with reſpect to the ſecond equation, at laſt obtain the values of all the ſeveral roots of the ſaid firſt equation, to as great a degree of exactneſs as we pleaſe. And this ſeems to be all that is neceſſary to be done towards making this method of reſolving equations by approximation, according to the directions given us by Mr. Raphſon, quite compleat. I therefore now hope that, after the careful peruſal of all the foregoing Tracts contained in this volume, the reader will find himſelf perfectly maſter of Mr. Raphſon's method of reſolving high affected equations by approximation, and will be able to apply it readily to the reſolution of any Algebraick equation that may be propoſed to him;—to contribute to which was the principal object I had in view in preparing theſe diſcourſes for the preſs. I might therefore here with propriety put an end to this volume. But, as there is another method of reſolving all equations by approximation, which was invented by the celebrated Vieta, above a hundred years before the publication of Mr. Raphſon's and Dr. Halley's methods, and which was generally

generally adopted and practiſed by the ableſt Mathematicians of thoſe times, and particularly by Mr. Oughtred, Dr. Pell, and Dr. Wallis, though it has ſince been abandoned on account of the greater facility of the methods of Sir Iſaac Newton, Mr. Raphſon, and Dr. Halley;—I conceive that it will be agreeable to the readers of theſe Tracts to be made acquainted with the principles on which it is grounded, and the manner in which we are enabled by it to inveſtigate the value of the root of any equation whatſoever. And for this reaſon I ſhall now proceed to lay before the reader a ſpecimen of this method, by reſolving in the manner preſcribed by it one of the affected equations which Vieta himſelf has exhibited as an example of it: but in doing this I ſhall not adopt that writer's language, or phraſeology, becauſe it is now grown ſo obſolete that the uſe of it would only tend to increaſe the difficulty of underſtanding the method itſelf.

Art. 2. The equation I ſhall ſelect for this purpoſe is that which is the ſubject of the 15th Problem of Vieta's Diſcourſe, intitled, "*De Numeroſâ Poteſtatum Adfectarum Reſolutione*," in pages 208, 209, and 210 of Schooten's edition of his works publiſhed at Leyden in the year 1646, and which is expreſſed as follows in his notation; to wit, $1QC - 5C + 500N = 7{,}905{,}504$; that is, "the product of the multiplication of the ſquare (Q) of a certain unknown number (N) into the cube (C) of the ſame number, being firſt diminiſhed by ſubtracting from it 5 times the cube (C) of the ſaid number, and being then increaſed by adding to the remainder of the

ſaid ſubtraction 500 times the ſaid number (N) itſelf; and the reſult being ſuppoſed to be equal to the given, or known, number 7,905,504; it is required to find the ſaid number (N) itſelf." This problem, if expreſſed in our preſent more convenient Algebraick notation, will, if we put x for the unknown number ſought, be as follows: "If $x^5 - 5x^3 + 500x$ be $= 7,905,504$, it is required to find the number x." We will therefore now endeavour to ſhew in what manner Vieta proceeds to find the ſaid number x, or the root of the equation $x^5 - 5x^3 + 500x = 7,905,504$.

Art. 3. The trinomial quantity $x^5 - 5x^3 + 500x$, (which forms the firſt, or left-hand, ſide of the equation $x^5 - 5x^3 + 500x = 7,905,504$) is called by Vieta *an affected fifth power of* x, in contra-diſtinction to the ſingle quantity x^5 which he calls *the pure fifth power of* x; and he calls the ſaid trinomial quantity by this name of *an affected fifth power of* x, becauſe the quantity x^5, or the pure fifth power of x, occurs in it, but is intermixed with, and *affected*, or *altered in it's magnitude, by*, the two terms $5x^3$ and $500x$, which involve two lower powers of x, of which the former term $5x^3$ is ſubtracted from x^5, and therefore tends to diminiſh it, and the latter term $500x$ is added to x^5, and therefore tends to increaſe it. And Vieta's manner of inveſtigating the value of x in the ſaid trinomial quantity, or affected fifth power of x, to wit, $x^5 - 5x^3 + 500x$, when the ſaid fifth power is ſuppoſed to be equal to the known number 7,905,504, is ſimilar to, and derived from, his manner of inveſtigating the value of x in the pure fifth power of x, to wit, the

ſingle

ſingle quantity x^5, when the ſaid pure fifth power of x, or the ſingle quantity x^1, is equal to the ſame known number 7,905,504; or, in other words, his manner of reſolving the affected equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ is ſimilar to, and derived from, his manner of reſolving the pure equation $x^5 = 7{,}905{,}504$, or of extracting the fifth root of the known number 7,905,504. Therefore, in order to underſtand Vieta's manner of inveſtigating the value of x in the affected equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$, we muſt firſt conſider his manner of inveſtigating the value of x in the pure equation $x^5 = 7{,}905{,}504$, or his manner of extracting the 5th root of the number 7,905,504. Now his manner of extracting this 5th root is as follows:

The Extraction of the Fifth Root of the Number 7,905,504, *according to Vieta's Method of Inveſtigation.*

Art. 4. We muſt firſt compare the given number 7,905,504, (of which we are required to find the 5th root,) with the fifth power of the number 10, in order to determine whether the ſaid 5th root, which we are ſeeking, will be greater, or leſs, than 10.

Now the fifth power of 10 is 100,000, which is much leſs than the given number 7,905,504. Therefore 10 will

will be leſs than the fifth root of the ſaid given number, or the ſaid fifth root will be greater than 10.

We muſt next compare the given number 7,905,504 with the fifth power of the number 100, in order to determine whether the ſaid fifth root of 7,905,504, which we are ſeeking, will be greater, or leſs, than 100.

Now the fifth power of 100 is 10,000,000,000, which is much greater than the given number 7,905,504. Therefore 100 will be greater than the fifth root of the ſaid given number, or the ſaid fifth root will be leſs than 100.

We therefore now know that the ſaid fifth root, which we are ſeeking, is greater than 10, but leſs than 100.

But the higheſt figure of every number that is greater than 10 but leſs than 100 muſt be a figure in the place of tens, and muſt be either 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the place of tens, and conſequently muſt be equal to either 10, or 20, or 30, or 40, or 50, or 60, or 70, or 80, or 90.

Therefore the higheſt figure of the number which is the fifth root of 7,905,504, will be a figure in the place of tens, and will be either 1, 2, 3, 4, 5, 6, 7, 8, or 9 in the place of tens, and conſequently will be equal either to 10, or 20, or 30, or 40, or 50, or 60, or 70, or 80, or 90.

Further,

Further, the number 7,905,504 is greater than 3,200,000, or than 32 × 100,000, or than $\overline{2}|^5 \times \overline{10}|^5$, but leſs than 24,300,000, or than 243 × 100,000, or than $\overline{3}|^5 \times \overline{10}|^5$. Therefore the fifth root of 7,905,504 will be greater than the fifth root of 3,200,000, or than 2 × 10, or than 20; but it will be leſs than the fifth root of 24,300,000, or than 3 × 10, or than 30. Therefore the higheſt figure of the number which is the fifth root of 7,905,504 will be a 2 in the place of tens, and will repreſent, or ſtand for, 20. And thus we have found the firſt, or higheſt, figure of the fifth root of 7,905,504, which we are ſeeking. Q. E. I.

Art. 5. Now let a be put for 20, and z for the unknown remaining part of the fifth root of 7,905,504.

Then will $a + z$ be $= \sqrt[5]{7,905,504}$, and conſequently $\overline{a+z}|^5$ will be $= 7,905,504$.

But $\overline{a+z}|^5$ is $= a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$.

Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504$; and (ſubtracting a^5 from both ſides,) $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504 - a^5$ $(= 7,905,504 - \overline{20}|^5 = 7,905,504 - 3,200,000) = 4,705,504$, which Vieta calls *the Reſolvend.* Therefore $5a^4z$ alone will be leſs than 4,705,504, and conſequently z will be leſs than $\frac{4,705,504}{5a^4}$, or than $\frac{4,705,504}{5 \times \overline{20}|^4}$, or than $\frac{4,705,504}{5 \times 160,000}$,

or

or than $\frac{4,705,504}{800,000}$, or than 5.8 &c. We will therefore ſuppoſe z (which we know to be leſs than 5.8 &c,) to be = 5, and will ſubſtitute 5 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, in order to diſcover whether the value of that quinquinomial quantity reſulting from ſuch ſubſtitution will be equal to, greater than, or leſs than, 4,705,504, and conſequently whether 5 will be equal to, greater than, or leſs than, the true value of z.

Now, if z is = 5, we ſhall have $z^2 = 25$, and $z^3 = 125$, and $z^4 = 625$, and $z^5 = 3125$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ ($= 5a^4 \times 5 + 10a^3 \times 25 + 10a^2 \times 125 + 5a \times 625 + 3125 = 5 \times 160,000 \times 5 + 10 \times 8000 \times 25 + 10 \times 400 \times 125 + 5 \times 20 \times 625 + 3125 = 800,000 \times 5 + 80,000 \times 25 + 4000 \times 125 + 100 \times 625 + 3125 = 4,000,000 + 2,000,000 + 500,000 + 62,500 + 3125$) = 6,565,625. This number is greater than 4,705,504. Therefore 5 muſt be greater than z, or z (which we before knew to be leſs than 5.8 &c) will not only be leſs than 5.8 &c, but alſo leſs than 5.

Let us therefore in the next place ſuppoſe z to be = 4, and try the effect of that ſuppoſition.

Now, if z is = 4, we ſhall have $z^2 = 16$, and $z^3 = 64$, and $z^4 = 256$, and $z^5 = 1024$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 4 + 10a^3 \times 16 + 10a^2 \times 64 + 5a \times 256 + 1024$ ($= 5$

$\times$ 160,000 $\times$ 4 + 10 $\times$ 8000 $\times$ 16 + 10 $\times$ 400 $\times$ 64 + 5 $\times$ 20 $\times$ 256 + 1024 = 800,000 $\times$ 4 + 80,000 $\times$ 16 + 4000 $\times$ 64 + 100 $\times$ 256 + 1024 = 3,200,000 + 1,280,000 + 256,000 + 25600 + 1024) = 4,762,624. This number is alſo greater than 4,705,504, though not in any great proportion. Therefore 4 muſt be ſomewhat greater than z, or z muſt be leſs than 4, as well as leſs than 5.8 &c and 5. We will therefore in the next place ſuppoſe z to be = 3, and try the effect of that ſuppoſition.

Now, if z is = 3, we ſhall have $z^2 = 9$, and $z^3 = 27$, and $z^4 = 81$, and $z^5 = 243$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 3 + 10a^3 \times 9 + 10a^2 \times 27 + 5a \times 81 + 243$ (= 800,000 $\times$ 3 + 80,000 $\times$ 9 + 4000 $\times$ 27 + 100 $\times$ 81 + 243 = 2,400,000 + 720,000 + 108,000 + 8,100 + 243) = 3,236,343. This number is leſs than 4,705,504. Therefore 3 is leſs than z, and conſequently z (though leſs than 4,) is greater than 3. Therefore the firſt, or higheſt, figure of the value of z will be a 3 in the place of units. Therefore $a + z$ will be greater than $a + 3$, or 20 + 3, or 23, but leſs than $a + 4$, or 20 + 4, or 24; and therefore 23 will be the two higheſt figures of the value of $a + z$, or of the fifth root of the propoſed number 7,905,504. Q. E. I.

Art. 6. If more figures of the value of the fifth root of 7,905,504 are wanted, we muſt proceed as follows:

When z is = 3, or $a + z$ is = 23, we have ſeen that the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4$

or than $\frac{4,705,504}{800,000}$, or than 5.8 &c. We will therefore ſuppoſe z (which we know to be leſs than 5.8 &c,) to be = 5, and will ſubſtitute 5 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, in order to diſcover whether the value of that quinquinomial quantity reſulting from ſuch ſubſtitution will be equal to, greater than, or leſs than, 4,705,504, and conſequently whether 5 will be equal to, greater than, or leſs than, the true value of z.

Now, if z is = 5, we ſhall have $z^2 = 25$, and $z^3 = 125$, and $z^4 = 625$, and $z^5 = 3125$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ ($= 5a^4 \times 5 + 10a^3 \times 25 + 10a^2 \times 125 + 5a \times 625 + 3125 = 5 \times 160,000 \times 5 + 10 \times 8000 \times 25 + 10 \times 400 \times 125 + 5 \times 20 \times 625 + 3125 = 800,000 \times 5 + 80,000 \times 25 + 4000 \times 125 + 100 \times 625 + 3125 = 4,000,000 + 2,000,000 + 500,000 + 62,500 + 3125$) = 6,565,625. This number is greater than 4,705,504. Therefore 5 muſt be greater than z, or z (which we before knew to be leſs than 5.8 &c) will not only be leſs than 5.8 &c, but alſo leſs than 5.

Let us therefore in the next place ſuppoſe z to be = 4, and try the effect of that ſuppoſition.

Now, if z is = 4, we ſhall have $z^2 = 16$, and $z^3 = 64$, and $z^4 = 256$, and $z^5 = 1024$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 4 + 10a^3 \times 16 + 10a^2 \times 64 + 5a \times 256 + 1024$ ($= 5$

$\times$ 160,000 $\times$ 4 + 10 $\times$ 8000 $\times$ 16 + 10 $\times$ 400 $\times$ 64 + 5 $\times$ 20 $\times$ 256 + 1024 = 800,000 $\times$ 4 + 80,000 $\times$ 16 + 4000 $\times$ 64 + 100 $\times$ 256 + 1024 = 3,200,000 + 1,280,000 + 256,000 + 25600 + 1024) = 4,762,624. This number is alſo greater than 4,705,504, though not in any great proportion. Therefore 4 muſt be ſomewhat greater than z, or z muſt be leſs than 4, as well as leſs than 5.8 &c and 5. We will therefore in the next place ſuppoſe z to be = 3, and try the effect of that ſuppoſition.

Now, if z is = 3, we ſhall have $z^2 = 9$, and $z^3 = 27$, and $z^4 = 81$, and $z^5 = 243$, and conſequently $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 5a^4 \times 3 + 10a^3 \times 9 + 10a^2 \times 27 + 5a \times 81 + 243$ (= 800,000 $\times$ 3 + 80,000 $\times$ 9 + 4000 $\times$ 27 + 100 $\times$ 81 + 243 = 2,400,000 + 720,000 + 108,000 + 8,100 + 243) = 3,236,343. This number is leſs than 4,705,504. Therefore 3 is leſs than z, and conſequently z (though leſs than 4,) is greater than 3. Therefore the firſt, or higheſt, figure of the value of z will be a 3 in the place of units. Therefore $a + z$ will be greater than $a + 3$, or 20 + 3, or 23, but leſs than $a + 4$, or 20 + 4, or 24; and therefore 23 will be the two higheſt figures of the value of $a + z$, or of the fifth root of the propoſed number 7,905,504. Q. E. I.

Art. 6. If more figures of the value of the fifth root of 7,905,504 are wanted, we muſt proceed as follows:

When z is = 3, or $a + z$ is = 23, we have ſeen that the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4$

$5az^4 + z^5$ is $= 3,236,343$. But a^5 is $(= \overline{20}|^5) = 3,200,000$. Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 3,200,000 + 3,236,343 = 6,436,343$; that is, $\overline{a+z}|^5$, or $\overline{23}|^5$, will be $= 6,436,343$.

Now let a denote 23, inſtead of 20; and let z be put for the difference between 23 and the true value of the 5th root of the propoſed number 7,905,504.

Then will $\overline{a+z}|^5$ be $= 7,905,504$.

But $\overline{a+z}|^5$ is $= a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$.

Therefore $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504$.

But a^5, or $\overline{23}|^5$, is $= 6,436,343$.

Therefore $6,436,343 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504$; and conſequently the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7,905,504 - 6,436,343 = 1,469,161$. Therefore $5a^4z$, or the firſt term of the ſaid quinquinomial quantity, will be leſs than 1,469,161, and conſequently z will be leſs than $\frac{1,469,161}{5a^4}$, or than $\frac{1,469,161}{5 \times \overline{23}|^4}$, or than $\frac{1,469,161}{5 \times 279,841}$, or than $\frac{1,469,161}{1,399,205}$, or than 1.05.

We

We will therefore ſuppoſe z to be $= 1$, and will ſubſtitute 1 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z + 10a^2z^3 + 5az^4 + z^5$ (which is equal to 1,469,161,) in order to diſcover whether the reſult will be equal to, greater than, or leſs than, 1,469,161, and conſequently whether 1 will be equal to, greater than, or leſs than, the true value of z in the ſaid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or the true exceſs of the 5th root of 7,905,504 above the number 23.

Now, if z is $= 1$, we ſhall have $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ ($= 5a^4 \times 1 + 10a^3 \times 1 + 10a^2 \times 1 + 5a \times 1 + 1 = 5a^4 + 10a^3 + 10a^2 + 5a + 1 = 5 \times \overline{23}^4 + 10 \times \overline{23}^3 + 10 \times \overline{23}^2 + 5 \times 23 + 1 = 5 \times 279,841 + 10 \times 12,167 + 10 \times 529 + 5 \times 23 + 1 = 1,399,205 + 121,670 + 5290 + 115 + 1) = 1,526,281$. This number is greater than 1,469,161; and conſequently 1 is greater than the true value of z in the ſaid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or than the exceſs of the fifth root of 7,905,504 above the number 23. This alſo might have been concluded from what was ſhewn in art. 5, to wit, that this fifth root would be leſs than 24, though greater than 23.

We will therefore now ſuppoſe z to be $= 0.9$, and will ſubſtitute 0.9 inſtead of z in the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ (which is equal to 1,469,161,) in order to diſcover whether the reſult will be equal to, greater than, or leſs than, 1,469,161, and conſequently whether 0.9 will be equal to, greater

greater than, or leſs than, the true value of z in the ſaid equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, or the true exceſs of the fifth root of 7,905,504 above the number 23.

Now, if z is $= 0.9$, we ſhall have $z^2 = 0.81$, and $z^3 = 0.729$, and $z^4 = 0.6561$, and $z^5 = 0.590,49$. Therefore $5a^4z$ will be $(= 5a^4 \times 0.9 = 5 \times \overline{23}^4 \times 0.9 = 5 \times 279,841 \times 0.9 = 1,399,205 \times 0.9) = 1,259,284.5$, and $10a^3z^2$ will be $(= 10 \times a^3 \times 0.81 = 10 \times \overline{23}^3 \times 0.81 = 10 \times 12,167 \times 0.81 = 121,670 \times 0.81) = 98,552.70$, and $10a^2z^3$ will be $(= 10 \times a^2 \times 0.729 = 10 \times \overline{23}^2 \times 0.729 = 10 \times 529 \times 0.729 = 5290 \times 0.729) = 3856.410$, and $5az^4$ will be $(= 5 \times 23 \times 0.6561 = 115 \times 0.6561) = 75.4515$, and conſequently the whole quinquinomial quantity $5a^4z + + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $(= 1,259,284.5 + 98,552.70 + 3856.410 + 75.4515 + 0.590,49) = 1,361,769.651,99$; which is leſs than 1,469,161. Therefore 0.9 will be leſs than the true value of z in the equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 1,469,161$, and conſequently z will be greater than 0.9, though leſs than 1. Therefore 0.9 will be the firſt, or higheſt, figure of the true value of z, and conſequently 23.9 will be the three firſt, or higheſt, figures of the value of $a + z$, or of the fifth root of the propoſed number 7,905,504. Q. E. I.

Art. 7. If more figures ſtill of the true value of the fifth root of 7,905,504 are wanted, we muſt now put $a = 23.9$, and $a + z = \sqrt[5]{7,905,504}$, and conſequently

quently $\overline{a + z}^5 = 7{,}905{,}504$, or $a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 7{,}905{,}504$; whence it will follow that the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ will be $= 7{,}905{,}504 - a^5 = 7{,}905{,}504 - \overline{23.9}^5$ $(= 7{,}905{,}504 - \overline{23 + 0.9}^5 = 7{,}905{,}504 - \overline{23}^5 - 5 \times \overline{23}^4 \times 0.9 - 10 \times \overline{23}^3 \times 0.81 - 10 \times \overline{23}^2 \times 0.729 - 5 \times 23 \times 0.9 - 0.590{,}49 = 7{,}905{,}504 - \overline{23}^5 - 1{,}361{,}769 = 7{,}905{,}504 - 6{,}436{,}343 - 1{,}361{,}769.651{,}99 = 1{,}469{,}161 - 1{,}361{,}769.651{,}09) = 107{,}391.348{,}01$. Therefore $5a^4z$ alone will be leſs than 107,391.348,01, and conſequently z will be leſs than $\frac{107{,}391.348{,}01}{5a^4}$, or than $\frac{107{,}391.348{,}01}{5 \times 23.9^4}$, or than $\frac{107{,}391.348{,}01}{5 \times 326{,}280.8641}$, or than $\frac{107{,}391.348{,}01}{1{,}631{,}404.3205}$, or than 0.06 &c. Therefore the firſt, or higheſt, figure of the true value of z muſt be either 0.06, that is, 6 in the ſecond place of decimal fractions, or ſome lower figure; and, if the value of the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$ reſulting from the ſubſtitution of 0.06 in it's terms inſtead of z, ſhall be equal to, or leſs than, 107,391.348,01, (the abſolute term of the laſt equation $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 = 107{,}391.348{,}01$,) the ſaid fraction 0.06 will at the ſame time be either exactly equal to, or leſs than, the true value of z, and in the latter caſe the figure 6, in the ſecond place of decimal fractions, will be the firſt, or higheſt, figure of the ſaid true value

by putting a for the part of the root that is already diſcovered, and x for the unknown remainder of it, and proceeding in the manner that has been deſcribed in the foregoing articles; every new proceſs of the computation giving us a new figure of the ſaid root.

A SCHOLIUM.

Art. 8. THIS method of extracting the fifth root of the number 7,905,504 is perfectly juſt and accurate, but is, as we have ſeen, attended with a great deal of labour. And this has induced the Mathematicians of the latter part of the laſt century to look-out for eaſier methods of obtaining the ſame end. And in this they have been remarkably ſucceſsfull, partly by means of the noble invention of Logarithms, (found out by Lord Napier in the year 1614, and brought to great perfection by Mr. Henry Briggs in the year 1624,) and partly by Mr. Raphſon's and Monſieur de Lagny's methods of extracting the roots of numbers by approximation. For by the common Tables of Logarithms we may find any root of a given number exact to five places of figures with very little trouble, without making uſe of the proportional parts that are ſet down in theſe Tables; and by the help of thoſe proportional parts we may obtain theſe roots to two figures more, that is, to ſeven places of figures, or, at leaſt, to ſix places of figures. And, when we have by this, or any other, means found the root of a number exact to any number of figures, we

may, by only one application of the expreſſion given us by *Monſieur de Lagny* for finding a more exact value of it, find twice as many more figures, or, at leaſt, twice as many more figures wanting one, of the ſaid root, as we had before obtained by means of the Table of Logarithms, or in any other way; ſo that, if the root already found (by means of the Table of Logarithms, or otherwiſe) was exact to five places of figures, we may find, by that expreſſion of Monſieur de Lagny, ten more figures, or, at leaſt, nine more figures, of the ſaid root, that will be all exact. Of this I will now give an example in the caſe of the fifth root of the number 7,905,504, which we have juſt now extracted by the method of Vieta, and of which we have found the firſt four figures to be 23.96.

The Extraction of the Fifth Root of the ſame Number, 7,905,504, by Means of the Expreſſion given by Monſieur de Lagny for that Purpoſe.

Art. 9. Monſieur de Lagny's general expreſſion for the value of any root of a given number, when a firſt near value of the ſaid root, that is leſs than it's true value, has been already found, is as follows: If the given number be called N, and the mth root of the ſaid number be required to be found, m being any whole number whatſoever; and a be a firſt near value of the root ſought, that is leſs than it's true value; the more accurate value of the ſaid mth root of the number N will be very nearly equal to $a + \frac{2a \times N - a^m}{\overline{m-1} \times N + \overline{m+1} \times a^m}$.

Now

Now let it be required to find the fifth root of the number 7,905,504. And let it be ſuppoſed that we have already found, (either by a Table of Logarithms, or in ſome other way,) that the three firſt figures of this fifth root are 23.9.

Then will N be = 7,905,504, and m will be = 5, and a will be = 23.9, and a^m will be $(= \overline{23.9}^5) =$ 7,798,112.651,99, and $N - a^m$ will be $(=$ 7,905,504 − 7,798,112.651,99) = 107,391.348,01, and $2a$ will be $(= 2 \times 23.9) = 47.8$, and $2a \times \overline{N - a^m}$ will be $(=$ 47.8 × 107,391.348,01) = 5,133,306.434,878. And $m - 1$ will be $(= 5 - 1) = 4$, and $m + 1$ will be $(= 5 + 1) = 6$, and $\overline{m - 1} \times N$ will be $(=$ 4 × 7,905,504) = 31,622,016, and $\overline{m + 1} \times a^m$ will be (= 6 × 7,798,112.651,99) = 46,788,675.911,94, and $\overline{m - 1} \times N + \overline{m + 1} \times a^m$ will be (= 31,622,016 + 46,788,675.911,94) = 78,410,691.911,94. Therefore $\frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ will be $= \frac{5,133,306,434,878}{78,410,691.911,94}$ = 0.065,466,9. Therefore $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ will be = 23.9 + 0.065,466,9, or 23.965,466,9; that is, the fifth root of the number 7,905,504 will be very nearly = 23.965,466,9. Q. E. I.

The Extraction of the Fifth Root of the ſame Number, 7,905,504, by Means of the Expreſſion given by Mr. Raphſon for that Purpoſe.

Art. 10. Mr. Raphſon's expreſſion for the ſecond near value of the *m*th root of the number N is $a + \frac{N - a^m}{ma^{m-1}}$; which in the preſent caſe is $= 23.9 + \frac{7,905,504 - (23.9)^5}{5 \times (23.9)^4}$ $(= 23.9 + \frac{7,905,504 - 7,798,112.651,99}{5 \times 326,280.8641} = 23.9 + \frac{107,391.348,01}{1,631,404.3205} = 23.9 + 0.0658) = 23.9658$. Therefore, according to this expreſſion given us by Mr. Raphſon, the 5th root of the number 7,905,504 is $= 23.9658$. Q. E. I.

This value of $\sqrt[5]{7,905,504}$ agrees with the former value of it found above by Monſieur de Lagny's more accurate expreſſion, to wit, 23.965,466,9, in the five firſt figures 23.965, which therefore muſt be exact: and conſequently this ſimple expreſſion of Mr. Raphſon has given us two new figures of the root ſought, to wit, the two figures 0.065, exactly.

Of the Reſemblance between Vieta's Method of extracting the ſaid Fifth Root of 7,905,504, *and Mr. Raphſon's Method of extracting it.*

Art. 11. Though, according to Vieta's method of proceeding in the extraction of theſe roots, we obtain but

but one new figure of the root by every new proceſs of the inveſtigation (which makes the extraction of a root to ten or twelve places of figures by this method intolerably laborious,) yet, in truth, each proceſs of the inveſtigation would, if we continued the diviſion of the known number that is equal to the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, or to 7,905,504 — a^5, (which known number Vieta calls *the reſolvend*,) to more than one figure in the quotient, give us, at leaſt, as many new figures of the root, wanting one figure, exact as there are figures in a, or the part of the root that is already known. Thus, for example, in the laſt proceſs of the foregoing inveſtigation, if we had continued the diviſion of the laſt dividend, or *reſolvend*, 107,391.348,01 by the diviſor $5a^4$, or 1,631,404,3205, (in art. 7,) to two figures in the quotient, (which would have been the two figures 0.065,) we ſhould have obtained the fifth figure of the true value of $a + z$, or of the 5th root of 7,905,504, to wit, 0.005, as well as the fourth figure of it, to wit, 0.06, and we might have concluded that 23.965 was the value of the ſaid fifth root to five places of figures, inſtead of concluding that 23.96 was it's value to only four places of figures. And this obſervation may, perhaps, have been the ground of Mr. Raphſon's very ſimple and uſefull method of extracting the roots of numbers by approximation, (which I have ſet forth and explained in pages 508, 509, and 510 of a Tract publiſhed in the year 1795 in a large volume in octavo, containing Mr. James Bernoulli's Doctrine of Permutations and Combinations, and other uſefull tracts on mathematical ſubjects,) the very ſame operation of diviſion, with the ſame dividend and diviſor, which is

here performed in one of the proceſſes of Vieta's method of extracting the roots of numbers, being performed alſo in Mr. Raphſon's method of extracting them, but with a continuation of the diviſion to more than one figure in the quotient, and, in general, to ſo many figures, wanting one, as there are figures in a, or the part of the root that is already known. But in the foregoing inveſtigation of the fifth root of the number 7,905,504, (contained in art. 4, 5, 6, and 7,) I have adhered ſtrictly to Vieta's method of proceeding, by which only one new figure of the root is obtained by each new proceſs of the inveſtigation; becauſe the only reaſon for my exhibiting in the foregoing articles this example of Vieta's method of extracting the roots of numbers, or reſolving equations involving only pure powers of an unknown quantity, was to lay a foundation for the more ready apprehenſion of his method of proceeding in the extraction of the roots of affected powers of an unknown quantity, or in the reſolution of affected equations. I therefore ſhall now endeavour to explain his method of reſolving the above-mentioned affected equation of the fifth order, to wit, $x^5 - 5x^3 + 500x = 7{,}905{,}504$.

A Reſolution of the Affected Equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ *in the Manner preſcribed by Vieta.*

Art. 12. Vieta begins his reſolution of this affected equation by obſerving that it's root muſt be nearly equal to the root of the pure equation $x^5 = 7{,}905{,}504$, which has the ſame known quantity, or abſolute term 7,905,504,

504, as the ſaid affected equation: and he thence concludes that the firſt, or higheſt, figure of the value of x in the ſaid affected equation will be a figure in the place of tens, and will be a 2, and conſequently will be equal to 20; becauſe the firſt, or higheſt, figure of the value of x in the pure equation $x^5 = 7,905,504$, or the firſt, or higheſt, figure of the fifth root of the number 7,905,504, is a figure in the place of tens and is a 2, and conſequently is equal to 20; as we have ſeen in the foregoing inveſtigation of it. Vieta then ſubſtitutes 20 inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$ (which he calls the fifth power of x affected by the ſubtraction of 5 times the cube of x from it, and by the addition of 500 times the firſt, or ſimple, power of x, or 500 times the unknown quantity x itſelf, to it,) in order to diſcover whether the reſult of ſuch ſubſtitution will be equal to, greater than, or leſs than, the abſolute term 7,905,504.

Now, if x be ſuppoſed to be $= 20$, we ſhall have $x^3 = 8000$, and $x^5 = 3,200,000$, and $5x^3$ $(= 5 \times 8000) = 40,000$, and $500x$ $(= 500 \times 20) = 10,000$. Therefore $x^5 - 5x^3 + 500x$ will be $(= 3,200,000 - 40,000 + 10,000 = 3,200,000 - 30,000) = 3,170,000$, which is leſs than 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$, or than the true value of the trinomial quantity $x^5 - 5x^3 + 500x$ in that equation. And hence he concludes that 20 muſt be leſs than the true value of x in that equation, and that the ſaid true value will be ſome number greater than 20, but leſs than 30.

Art. 13.

Art. 13. This concluſion will be confirmed by, ſecondly, ſuppoſing x to be = 30, and ſubſtituting 30 inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$. For, if x is = 30, we ſhall have $x^3 = 27{,}000$, and $x^5 = 24{,}300{,}009$, and $5x^3$ (= 5 × 27,000) = 135,000, and $500x$ (= 500 × 30) = 15,000, and conſequently $x^5 - 5x^3 + 500x$ (= 24,300,000 — 135,000 + 15,000 = 24,300,000 — 120,000) = 24,180,000; which is much greater than 7,905,504. Therefore, while x increaſes from 20 to 30, the trinomial quantity $x^5 - 5x^3 + 500x$ will increaſe from 3,170,000 to 24,180,000; and conſequently there muſt have been a point of time, during the increaſe of x from 20 to 30, at which the ſaid trinomial quantity will have been exactly equal to any quantity greater than 3,170,000 and leſs than 24,180,000, and conſequently to the quantity 7,905,504, or the abſolute term of the equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$; or there will be ſome quantity greater than 20, but leſs than 30, that, being ſubſtituted inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$, will make the ſaid trinomial quantity be exactly equal to 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$; or, in other words, the root of the ſaid equation will be greater than 20, but leſs than 30. Q. E. D.

Art. 14. Having thus found that x, or the root of the equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$, is greater than 20, but leſs than 30, Vieta proceeds to inveſtigate the addition that muſt be made to 20 in order to make it equal

equal to x, or the root of the ſaid equation. And his method of making this inveſtigation is in ſubſtance equivalent to the following train of reaſoning, though expreſſed in other words.

Let the letter a be put for 20, or the part of the root x that is already known; and let z be put for the unknown remainder of the ſaid root.

Then will x be $= a + z$, and conſequently x^5 will be $(= \overbrace{a+z}^{5}) = a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, and x^3 will be $(= \overbrace{a+z}^{3}) = a^3 + 3a^2z + 3az^2 + z^3$, and conſequently $5x^3$ will be $(= 5 \times \overbrace{a^3 + 3a^2z + 3az^2 + z^3}) = 5a^3 + 15a^2z + 15az^2 + 5z^3$, and $500x$ will be $(= 500 \times \overbrace{a+z}) = 500a + 500z$. Therefore the trinomial quantity $x^5 - 5x^3 + 500x$ will be $=$ the multinomial quantity

$$\left\{\begin{array}{rrrrrr} a^5 & + 5a^4z & + 10a^3z^2 & + 10a^2z^3 & + 5az^4 & + z^5 \\ - 5a^3 & - 15a^2z & - 15az^2 & - 5z^3 & & \\ + 500a & + 500z. & & & & \end{array}\right\}$$

But the trinomial quantity $x^5 - 5x^3 + 500x$ is $= 7,905,504$.

Therefore the multinomial quantity

$$\left\{\begin{array}{rrrrrr} a^5 & + 5a^4z & + 10a^3z^2 & + 10a^2z^3 & + 5az^4 & + z^5 \\ - 5a^3 & - 15a^2z & - 15az^2 & - 5z^3 & & \\ + 500a & + 500z & & & & \end{array}\right\}$$

will alſo be $= 7,905,504$.

But

But $a^5 - 5a^3 + 500a$ has been ſhewn to be $= 3,170,000$; and $5a^4$ is $(= 5 \times \overline{20}^4 = 5 \times 160,000) = 800,000$, and $15a^2$ is $(= 15 \times 400) = 6000$, and conſequently $5a^4 - 15a^2 + 500$ is $(= 800,000 - 6000 + 500 = 794,000 + 500) = 794,500$; and $10a^3$ is $(= 10 \times 8000) = 80,000$, and $15a$ is $(= 15 \times 20) = 300$, and conſequently $10a^3 - 15a$ is $(= 80,000 - 300) = 79,700$; and $10a^2$ is $(= 10 \times 400) = 4000$, and conſequently $10a^2 - 5$ is $(= 4000 - 5) = 3995$; and $5a$ is $(= 5 \times 20) = 100$.

Therefore the multinomial quantity

$$\left\{\begin{array}{l} a^5 + 5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5 \\ - 5a^3 - 15a^2z - 15az^2 - 5z^3 \\ + 500a + 500z \end{array}\right\}$$

is $=$ the ſextinomial quantity $3,170,000 + 794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$.

Therefore the ſextinomial quantity $3,170,000 + 794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $= 7,905,504$. And conſequently (ſubtracting $3,170,000$ from both ſides,) the quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $(= 7,905,504 - 3,170,000) = 4,735,504$.

Therefore the ſingle term $794,500z$ will be leſs than $4,735,504$, and conſequently z will be leſs than $\frac{4,735,504}{794,500}$, or than 5.96 &c. We will therefore ſuppoſe

poſe z to be $= 5$, and will ſubſtitute 5 inſtead of z in the quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$, in order to diſcover whether the value of the ſaid quinquinomial quantity reſulting from this ſubſtitution will be equal to, greater than, or leſs than, 4,735,504, or the abſolute term of the laſt equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$, and conſequently whether 5 will be equal to, greater than, or leſs than, the true value of z in that equation.

Now, if z is $= 5$, we ſhall have $z^2 = 25$, and $z^3 = 125$, and $z^4 = 625$, and $z^5 = 3125$, and conſequently $794,500z$ $(= 794,500 \times 5) = 3,972,500$, and $79,700z^2$ $(= 79,700 \times 25) = 1,992,500$. Therefore $794,500z + 79,700z^2$ will be: $(= 3,972,500 + 1,992,500) = 5,965,000$, which is greater than 4,735,404. Therefore, *à fortiori*, the whole quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ muſt be greater than 4,735,504; and conſequently 5 muſt be greater than the true value of z in the ſaid equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$.

Q. E. I.

We will therefore in the next place ſuppoſe z to be $= 4$, and will ſubſtitute 4 inſtead of z in the terms of the ſaid quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$, in order to diſcover whether the value of the ſaid quinquinomial quantity reſulting from ſuch ſubſtitution will be equal to, greater than, or

leſs

leſs than, the number 4,735,504, or the abſolute term of the equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$, and conſequently whether 4 is equal to, greater than, or leſs than, the true value of z in that equation.

Now if z is $= 4$, we ſhall have $z^2 = 16$, and $z^3 = 64$, and $z^4 = 256$, and $z^5 = 1024$. Therefore $794,500z$ will be $(= 794,500 \times 4) = 3,178,000$, and $79,700z^2$ will be $(= 79,700 \times 16) = 1,275,200$, and $3995z^3$ will be $(= 3995 \times 64) = 255,680$, and $100z^4$ will be $(= 100 \times 256) = 25,600$, and conſequently the whole quinquinomial quantity $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$ will be $(= 3,178,000 + 1,275,200 + 255,680 + 25,600 + 1024) = 4,735,504$. Therefore 4 is exactly equal to the true value of z in the equation $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5 = 4,735,504$; and conſequently $a + z$, or $20 + z$, is exactly equal to $20 + 4$, or 24, or the root x of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$ is exactly equal to 24. Q. E. I.

Art. 15. And, accordingly, if we ſuppoſe x to be equal to 24, we ſhall have $xx\ (= \overline{24}^2) = 576$, and $x^3\ (= \overline{24}^3) = 13,824$, and $x^4\ (= \overline{24}^4) = 331,776$, and $x^5\ (= \overline{24}^5) = 7,962,624$, and $5x^3\ (= 5 \times 13,824) = 69,120$, and $500x\ (= 500 \times 24) = 12,000$, and conſequently $x^5 - 5x^3 + 500x\ (= 7,962,624 - 69,120 + 12,000 = 7,974,624 - 69,120) = 7,905,$

504.

504. Therefore 24 is the true value of x in the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$.

Q. E. D.

A Reſolution of the ſame Equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ *by Mr. Raphſon's Method of Approximation.*

Art. 16. I will now reſolve the foregoing equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ by Mr. Raphſon's method of approximation, that the circumſtances in which it agrees with Vieta's method of reſolving it, and thoſe in which the methods differ from each other, may be the more apparent. And, in order to obtain the value of a, or a firſt near value of the root x, to be made the ground, or baſis, of a further approach to it's true value in the manner preſcribed by Mr. Raphſon, it will be convenient to proceed as follows:

Let us, firſt, ſuppoſe x to be $= 10$, and try the effect of that ſuppoſition.

Now, if x is $= 10$, we ſhall have $x^3 = 1000$, and $x^5 = 100{,}000$, and $5x^3 = 5000$, and $500x$ $(= 500 \times 10) = 5000$, and conſequently $x^5 - 5x^3 + 500x$ $(= 100{,}000 - 5000 + 5000) = 100{,}000$; which is much leſs

leſs than 7,905,50;, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$.

We will therefore, in the ſecond place, ſuppoſe x to be $= 20$, and try the effect of this ſuppoſition.

Now, if x is $= 20$, we ſhall have $x^3 = 8000$, and $x^5 = 3{,}200{,}000$ and $5x^3$ $(= 5 \times 8000) = 40{,}000$, and $500x$ $(= 500 \times 20) = 10{,}000$, and conſequently $x^5 - 5x^3 + 500x = 3{,}200.000 - 40{,}000 + 10{,}000 = 3{,}210{,}000 - 40{,}000 = 3{,}170{,}000$; which is alſo leſs than 7,905,504, or the abſolute term of the propoſed equation.

We will therefore, in the third place, ſuppoſe x to be $= 30$, and try the effect of that ſuppoſition.

Now, if x is $= 30$, we ſhall have $x^3 = 27{,}000$, and $x^5 = 24{,}300{,}000$, and $5x^3$ $(= 5 \times 27{,}000) = 135{,}000$, and $500x$ $(= 500 \times 30) = 15{,}000$, and conſequently $x^5 - 5x^3 + 500x$ $(= 24{,}300{,}000 - 135{,}000 + 15{,}000 = 24{,}315{,}000 - 135{,}000) = 24{,}180{,}000$; which is greater than 7,905,504, or the abſolute term of the propoſed equation.

Therefore, while x increaſes from 20 to 30, the trinomial quantity $x^5 - 5x^3 + 500x$ will increaſe from 3,170,000 to 24,180,000, and conſequently will, at ſome one inſtant of time during the ſaid increaſe of x from 20 to 30, be equal to any quantity that is greater than 3,170,000 and leſs than 24,180,000, and therefore to the quantity

quantity 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$; or, there will be ſome quantity greater than 20, but leſs than 30, that, being ſubſtituted inſtead of x in the trinomial quantity $x^5 - 5x^3 + 500x$, will make the ſaid quantity be equal to 7,905,504; that is, the root of the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$ will be greater than 20, but leſs than 30.

Art. 17. We might now proceed to make further conjectures and trials concerning the magnitude of x, in order to approach a little nearer to it's true value than we have hitherto done, before we have recourſe to Mr. Raphſon's method of approximation to obtain a more exact value of it. But, as in Vieta's method of reſolving this equation, (which has been deſcribed in the preceeding articles,) the ſecond proceſs of the inveſtigation begins from a ſuppoſition that by the firſt proceſs of it we have only diſcovered that x is greater than 20, I will take 20 for the value of a, or the firſt near value of x, from which I will begin the further approach to it's true value by Mr. Raphſon's method; by which means the ſimilitude of the two methods to each other, and likewiſe the points in which they differ from each other, will be ſeen more diſtinctly.

I will therefore now ſuppoſe z to be equal to the exceſs of the true value of x above 20, or a, it's near value, and ſubſtitute $20 + z$, or $a + z$, inſtead of x, in the propoſed equation $x^5 - 5x^3 + 500x = 7{,}905{,}504$, omitting all the terms that involve any higher

powers of z than it's ſimple power, or z itſelf, agreeably to Mr. Raphſon's directions.

Now, if x is $= a + z$, we ſhall have $x^3 = \overline{a+z}^3 = a^3 + 3a^2z +$ &c, and $x^5 = \overline{a+z}^5 = a^5 + 5a^4z +$ &c, and $5x^3 \left(= 5 \times \overline{a^3 + 3a^2z + \&c}\right) = 5a^3 + 15a^2z +$ &c, and $500x \left(= 500 \times \overline{a+z}\right) = 500a + 500z$, and conſequently $x^5 - 5x^3 + 500x =$

$$\left\{\begin{array}{r} a^5 + 5a^4z + \&c \\ - 5a^3 - 15a^2z - \&c \\ + 500a + 500z \end{array}\right\} =$$

$$\left\{\begin{array}{r} 3{,}200{,}000 + 5 \times 160{,}090 \times z + \&c \\ - 5 \times 8000 - 15 \times 400 \times z - \&c \\ + 500 \times 20 + 500z \end{array}\right\} =$$

$$\left\{\begin{array}{r} 3{,}200{,}000 + 800{,}000 \times z + \&c \\ - 40{,}000 - 6000 \times z - \&c \\ + 10{,}000 + 500 \times z \end{array}\right\} =$$

$$\left\{\begin{array}{r} 3{,}210{,}000 + 800{,}500 \times z + \&c \\ - 40{,}000 - 6000 \times z - \&c \end{array}\right\}$$

$= 3{,}170{,}000 + 794{,}500 \times z$ &c.

But $x^5 - 5x^3 + 500x$ is $= 7{,}905{,}504$.

Therefore $3{,}170{,}000 + 794{,}500 \times z$ &c will alſo be $= 7{,}905{,}504$; and conſequently $794{,}500 \times z$ &c will be $(= 7{,}905{,}504 - 3{,}170{,}000) = 4{,}735{,}504$; that is,

794,500

794,500 × z together with the omitted terms included under the &c, will be = 4,735,504. Therefore 794,500 × z alone will be ſomewhat leſs than, but nearly equal to, 4,735,504; and conſequently z will be ſomewhat leſs than, but nearly equal to, $\frac{4,735,504}{794,500}$, or 5.96 &c. Therefore $a + z$, or $20 + z$, or the root of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$, will be ſomewhat leſs than, but nearly equal to, 20 + 5.96 &c, or 25.96 &c.

We will therefore ſuppoſe z to be equal to 5, and conſequently x, or $20 + z$, to be equal to 25, and try the effect of that ſuppoſition.

Now, if x is = 25, we ſhall have x^3 ($= \overline{25}^3$) = 15,625, and x^5 = 9,765,625, and $5x^3$ ($= 5 \times 15,625$) = 78,125, and $500x$ ($= 500 \times 25$) = 12,500, and conſequently $x^5 - 5x^3 + 500x$ ($= 9,765,625 - 78,125 + 12,500 = 9,778,125 - 78,125$) = 9,700,000; which is conſiderably greater than 7,905,504, or the abſolute term of the equation $x^5 - 5x^3 + 500x = 7,905,504$. Therefore 25 is greater than the true value of x in that equation.

We will therefore, in the next place, ſuppoſe z to be = 4, and conſequently $20 + z$, or x, to be = 20 + 4, or 24, and try the effect of that ſuppoſition.

Now, if x is = 24, we ſhall have x^3 ($= \overline{24}^3$) = 13,824, and x^5 ($= \overline{24}^5$) = 7,962,624, and $5x^3$ (= 5

$\times$ 13,824) = 69,120, and $500x$ ($= 500 \times 24$) = 12,000, and conſequently $x^5 - 5x^3 + 500x$ ($=$ 7,962,624 — 69,120 + 12,000 = 7,974,624 — 69,120) = 7,905,504, or the abſolute term of the propoſed equation $x^5 - 5x^3 + 500x = 7,905,504$. Therefore 24 is the true value of x in the ſaid equation.

Q. E. I.

A SCHOLIUM.

Art. 18. We may obſerve that in this application of Mr. Raphſon's method of approximation to the finding a ſecond near value of x after having already found that it was greater than 20, we came to the diviſion of the ſame dividend 4,735,504, by the ſame diviſor 794,500, by means of which we had obtained the value of z, or the difference between 20 and the true value of x in Vieta's inveſtigation; ſo that the reaſonings uſed in the two methods in this part of the inveſtigation are the ſame in ſubſtance, and produce the ſame final reſult, namely, a near value of z equal to the quotient, 5.96, of the diviſion of the number 4,735,504 by the number 794,500. But in Mr. Raphſon's method of proceeding the four latter terms of the quinquinomial quantity $5a^4z + 10a^3z^2 + 10a^2z^3 + 5az^4 + z^5$, or $794,500z + 79,700z^2 + 3995z^3 + 100z^4 + z^5$, are omitted, as unimportant to the

the final reſult, which is obtained by ſuppoſing the firſt of thoſe terms alone, to wit, the term $5a^4z$, or $794,500z$, to be equal to the number 4,735,504, and dividing the ſaid number (which Vieta calls *the reſolvend*, by $5a^4$, or 794,500, or the co-efficient of z. This omiſſion is a ſaving of unneceſſary labour in Mr. Raphſon's method of proceeding, and ſeems to be the only advantage it has over Vieta's method in the reſolution of the foregoing equation: But, when a, or the known part of the value of x in any equation, conſiſts of three, or four, or more, figures that are all true, or exact, Mr. Raphſon's method has a great advantage over Vieta's, which is owing to his continuing the quotient ariſing from the diviſion of the reſolvend by $5a^4$, or the co-efficient of z, (which diviſion occurs both in Vieta's method and in Mr. Raphſon's,) to more figures than Vieta does, namely, to as many figures, wanting one, as there are figures in a, or the part of the value of x that is already known; all which figures will be exact: whereas Vieta continues this diviſion to only one figure in the quotient, and ſo obtains but one new figure of the true value of x, or the root of the propoſed equation, by every new proceſs of his inveſtigation; which makes his method of reſolving theſe equations exceſſively laborious and tedious, when their roots are to be obtained to ten or twelve places of figures. If Vieta had happened to obſerve that theſe diviſions might be ſafely continued to ſeveral figures in the quotients, or that ſeveral figures in the quotients, (namely, as many, wanting one, as there are figures in a, or the part of the value of x that is already known,) would be exact, and had conſequently directed his readers to continue the

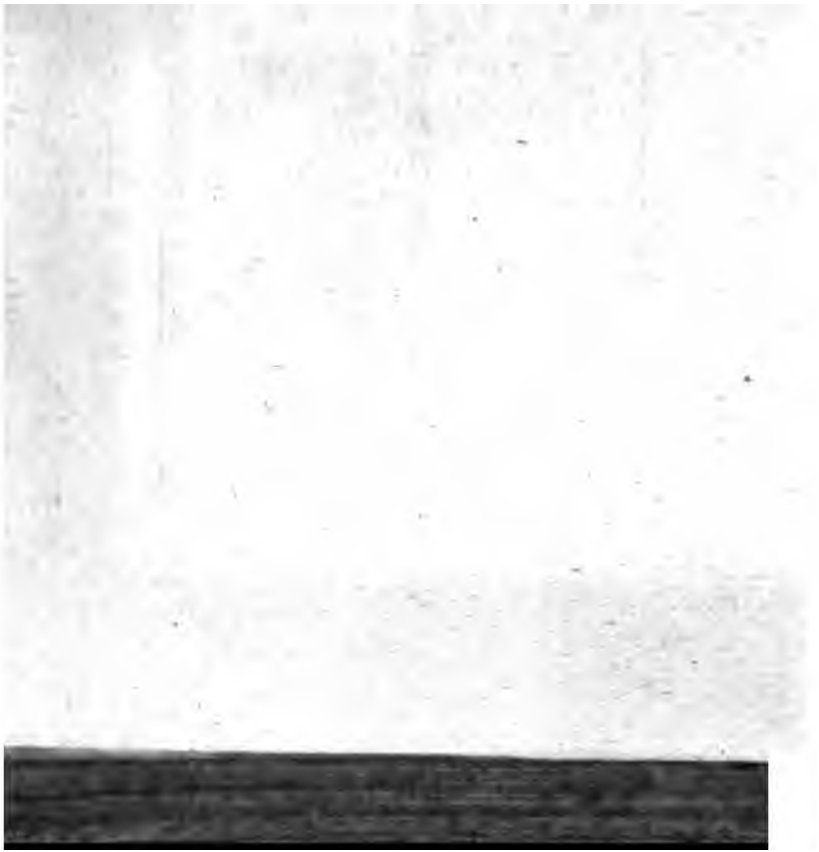
diviſions to that number of figures in the quotients, his method would, as I conceive, have been the very ſame with Mr. Raphſon's, and the art of reſolving equations of all kinds by approximation would not only have been invented by him, (as it has been,) but would have been brought by him at once to the higheſt degree of perfection of which it, probably, is capable. He therefore ought to be conſidered as the original founder of the whole doctrine of reſolving equations by approximation, or (as *Lucretius* ſays of his favourite philoſopher *Epicurus*,) as the *Pater et rerum Inventor* on this ſubject, though Sir Iſaac Newton, Mr. Raphſon, Monſieur de Lagny, and Dr. Halley, and perhaps ſome ſubſequent Mathematicians, have made valuable improvements on his method, by which the practice of it has been very much facilitated.

The End of the Specimen of Vieta's Method of reſolving Algebräick Equations.

REMARKS

ON

THE NUMBER OF *NEGATIVE* AND *IMPOSSIBLE* ROOTS IN ALGEBRÄICK EQUATIONS.



Remarks on the Number of Negative *and* Impoſſible *Roots in Algebräick Equations.*

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Article 1. SIR Iſaac Newton, Mr. George Campbell, the celebrated Mr. Mac Laurin of Edinburgh, the late Profeſſor Waring of Cambridge, and other eminent writers on Algebra have laid down rules for finding the number of impoſſible roots in any propoſed Algebräick equation; but theſe rules are far from being clear and ſatisfactory, and ſome of them are abſolutely unintelligible; which indeed is not to be wondered-at, ſince they are all founded on a falſe ſuppoſition, which vitiates all the concluſions derived from it. This ſuppoſition, (which they lay down as an indiſputable, and almoſt ſelf-evident, maxim,) is, "*That every Algebräick equation has as many roots as it has dimenſions,*" though in truth there are very few equations in which this maxim really takes place; to wit, only one ſingle form

form in each degree, or order, of equations. Thus, for example, of all the three forms of affected quadratick equations, to wit, $xx + px = q$, $xx - px = q$, and $px - xx = q$, it is only the third form, $px - xx = q$, that can ever have two roots, and *that* only when the abſolute term q is leſs than $\frac{pp}{4}$, or than the ſquare of $\frac{p}{2}$, or of half the co-efficient, p, of the unknown quantity x: and of all the thirteen different forms of affected cubick equations, to wit, $x^3 + px^2 = r$, $x^3 - px^2 = r$, $px^2 - x^3 = r$, $x^3 + qx = r$, $x^3 - qx = r$, $qx - x^3 = r$, $x^3 + px^2 + qx = r$, $x^3 + px^2 - qx = r$, $x^3 - px^2 + qx = r$, $x^3 - px^2 - qx = r$, $-x^3 + px^2 + qx = r$, $-x^3 + px^2 - qx = r$, and $-x^3 - px^2 + qx = r$, it is only the ninth form, $x^3 - px^2 + qx = r$, that can ever have three roots, and *that* only when the abſolute term r is leſs than $\frac{p^3}{27}$, or the cube of $\frac{p}{3}$, or of the third part of, p, the co-efficient of x^2: and of the forty-five different forms of affected biquadratick equations it is only the form $-x^4 + px^3 - qx^2 + rx = s$, or $rx - qx^2 + px^3 - x^4 = s$ that can ever have four roots, and *that* only when the abſolute term s is leſs than $\frac{p^4}{256}$, or the fourth power of $\frac{p}{4}$, or of the fourth part of, p, the co-efficient of x^3. Therefore, when theſe authors have laid down the foregoing general propoſition, (by which they extend what is true in only one form of every new degree, or order, of affected equations, and not

not always even in that one form, to all the other forms of equations of the ſame order,) they find themſelves under a neceſſity of giving ſpecious names to a parcell of quantities which they endeavour to make paſs for roots of theſe equations, though in truth they are not ſo, in order to cover the falſehood of their general propoſition, and give it, in words at leaſt, an appearance of truth; and with this view they call ſome of theſe quantities *negative roots* of the equation to which they relate, and others of them it's *impoſſible roots*. And to determine the number of the ſtrange quantities ſo denominated, and diſcover how many of the ſuppoſed roots of a propoſed Algebräick equation are *negative*, and how many are *impoſſible*, has been made by theſe myſterious writers an object of great importance and moſt ſubtle and profound inveſtigation.

Art. 2. That the aforeſaid general propoſition concerning the number of roots of an equation is falſe, will appear by examining only one form of an affected cubick equation that has all it's terms compleat; and that form ſhall be the ſimpleſt form that can be choſen, namely, that in which all the terms involving the unknown quantity x are added to each other, and their ſum is declared to be equal to a certain known quantity, to wit, the equation $x^3 + px^2 + qx = r$.

Now this equation evidently can have only one root. For, if any particular number, being ſubſtituted inſtead of x in the trinomial quantity $x^3 + px^2 + qx$, would make that quantity, (which is the ſum of the three terms

x^3,

x^3, px^2, and qx) become equal to the known quantity r, or, in other words, is the root of this equation, it is evident that any quantity greater, or leſs, than that number, being ſubſtituted in the ſaid trinomial quantity, would make it greater, or leſs, than it was before, and conſequently greater, or leſs, than the abſolute term r, or, in other words, would not be a root of the equation. Here therefore the myſterious Algebräiſts, who maintain that every cubick equation has three roots, find themſelves at a loſs to ſupport their aſſertion by aſſigning any poſſible quantities for the ſecond and third roots of the equation; and therefore they get out of the difficulty by declaring that this equation $x^3 + px^2 + qx = r$ has one real and poſitive, or affirmative, root, and *two impoſſible roots*. This is ſtrange language to be uſed in treating of Algebra, or *Univerſal Arithmetick*, which is, in itſelf, the plaineſt and cleareſt of all ſciences!

Art. 3. In the foregoing equation $x^3 + px^2 + qx = r$ we have ſeen how the myſterious Algebräiſts, in order to ſupport their fundamental, falſe, poſition, are obliged to have recourſe to the introduction of *impoſſible roots*. We will now take another form of cubick equation in which they get rid of their difficulties by the introduction of only *negative roots*, which ſeems to be a fiction ſomewhat leſs bold and abſurd, or, at leaſt, leſs ſhocking to the ear, than that of impoſſible roots. But it is ſtill a fiction, or falſe aſſertion: for it is introducing the roots of another equation, and declaring them to be roots of the equation under conſideration. Let us ſuppoſe the propoſed equation to be $px^2 + qx - x^3 = r$; which,

which, if the abſolute term r is leſs than a certain quantity, may have two roots, but never can have three. Here then theſe writers, to maintain their general poſition, "that every equation has as many roots as it has dimenſions," (according to which this cubick equation ought to have three roots,) find themſelves under a neceſſity of providing it with one root more; which they do in the following manner. They change the ſigns + and — of the terms qx and x^3, (which involve the odd powers of x,) into the contrary ſigns — and +, and thereby convert the original equation $px^2 + qx - x^3 = r$ into the equation $px^2 - qx + x^3 = r$, or $x^3 + px^2 - qx = r$, which is always poſſible, and has only one root; and this one root of this latter equation $x^3 + px^2 - qx = r$ they call a *negative root* of the former equation $px^2 + qx - x^3 = r$. And thus, with this *negative* root of the ſaid original equation $px^2 + qx - x^3 = r$ and it's two former roots, (which really belong to it, if r is leſs than a certain finite quantity,) they compleat the number of three roots which, according to their grand, fundamental, propoſition, it ought to have. And, if r ſhould be greater than the certain finite quantity juſt now alluded-to, and conſequently the original equation $px^2 + qx - x^3 = r$ ſhould become impoſſible, or, rather, falſe *, they then, to maintain their grand, fundamental, propo-

* For an equation is nothing more than a propoſition affirming the equality of one, or more, unknown quantities, or terms involving the powers of an unknown quantity, to a known quantity, and, therefore, when the former quantities cannot be equal to the latter, ought rather to be called *falſe* than *impoſſible*.

ſition,

ſition, (according to which this equation ought to have three roots,) declare that this equation has two impoſſible roots and one negative root.

Art. 4. By the *negative roots* of an equation we are therefore to underſtand the real roots, or, in the language of modern writers of Algebra, the poſitive, or affirmative, roots, of another equation which is derived from the propoſed equation by changing the ſigns + and — that are prefixed to thoſe terms of it which involve in them the odd powers of the unknown quantity x into the contrary ſigns; which *negative* roots of the propoſed equation will often be ſufficient in number to increaſe the number of it's real, or poſitive, or affirmative, roots to the number required by the above mentioned grand, fundamental, propoſition laid down by the myſterious Algebräiſts, to wit, the number of dimenſions of x in the higheſt term of the equation, or the number of units contained in the index of the higheſt power of x. And by *the impoſſible roots* of an equation we are to underſtand certain fictitious quantities which are not the roots of any equation whatſoever, and of which no clear and diſtinct idea can be formed, but which are in number equal to the exceſs of the number of units contained in the index of the higheſt power of x in the equation above the ſum of both the real (or poſitive, or affirmative) roots and the negative roots of the propoſed equation, and are therefore ſufficient to make the whole number of roots of the equation, real, or poſitive, negative, and impoſſible, be equal to the number of the dimenſions of the equation, or to the number of units

in the index of the higheſt power of x in the equation, agreeably to the grand, fundamental, maxim abovementioned that is ſo much inſiſted-on by modern Algebräiſts.

Art. 5. And hence it is evident that, to find the number of impoſſible roots belonging to any equation, we need only, firſt, aſcertain the number of it's real, or poſitive, roots, and, then that of it's *negative* roots, or of the real, or poſitive, roots of another equation that is derived from the firſt equation by changing the ſigns + and — that are prefixed to all thoſe of it's terms which involve any odd powers of x, and then add theſe two numbers together, and ſubtract their ſum from the number of the dimenſions of the equation, or the number of units in the index of the higheſt power of x in any of it's terms: for the remainder ariſing from this ſubtraction will be the number of the impoſſible roots of the equation.

See on this ſubject the foregoing part of this preſent Collection of Tracts, pages 285, 286, and 287.

THE END.

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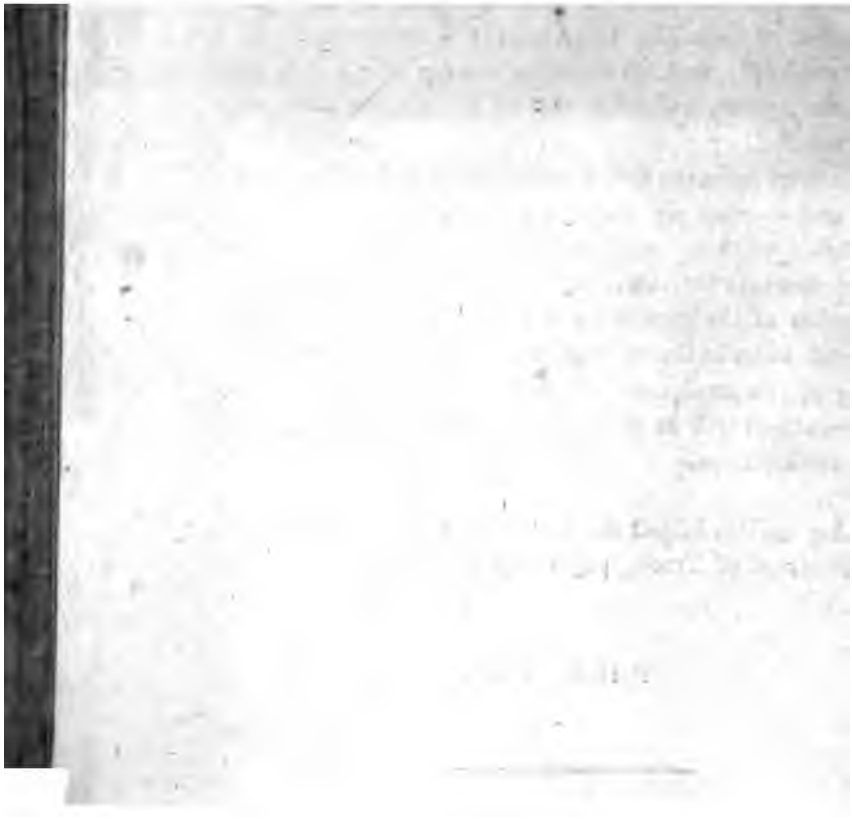

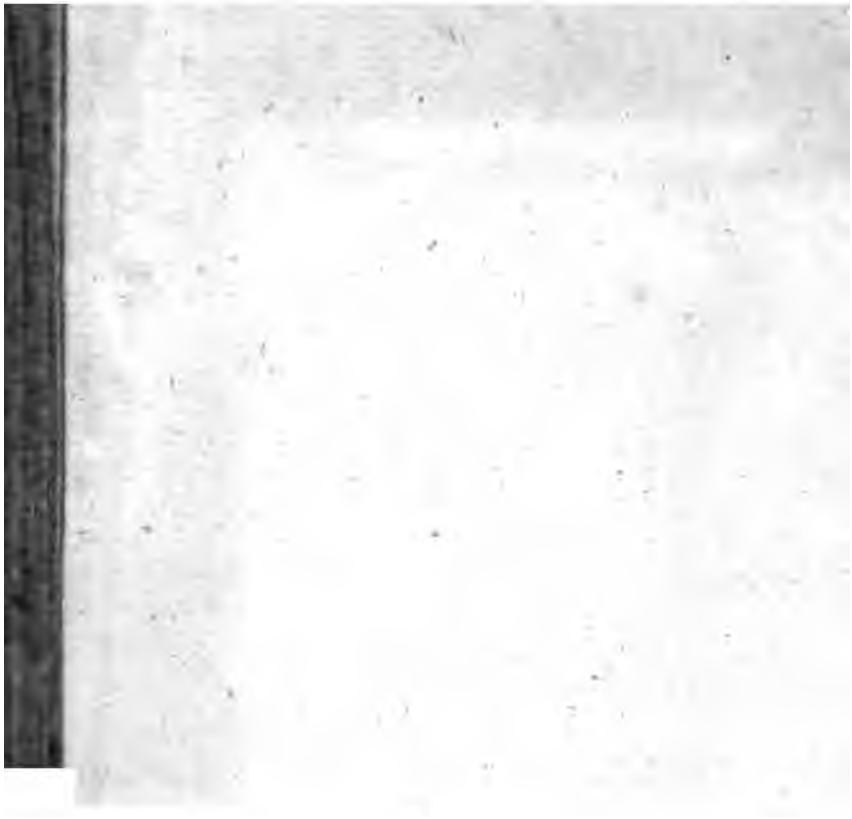

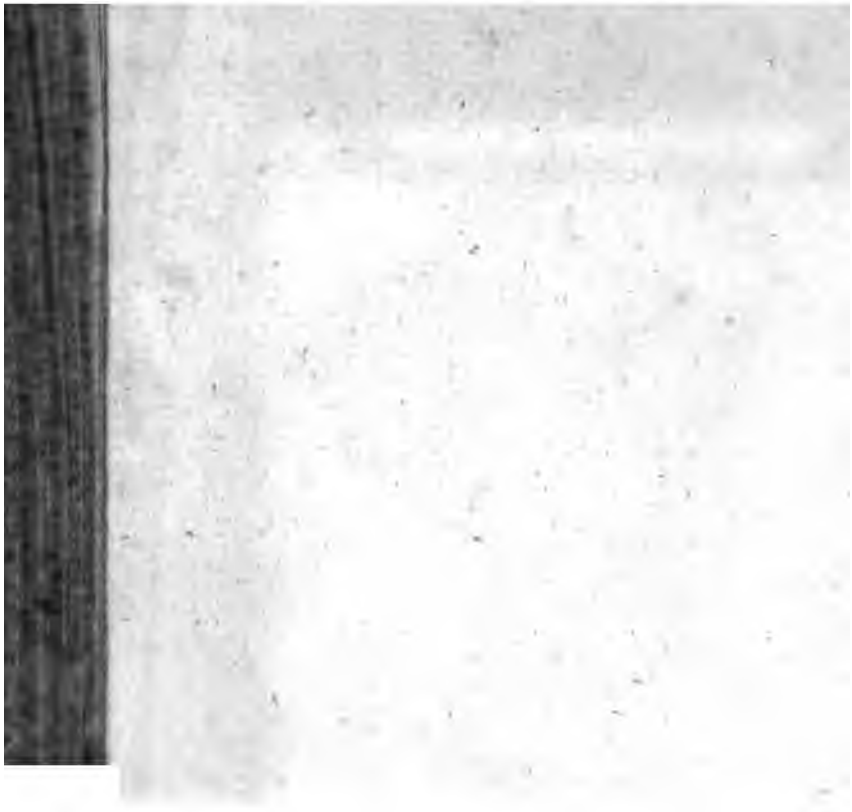

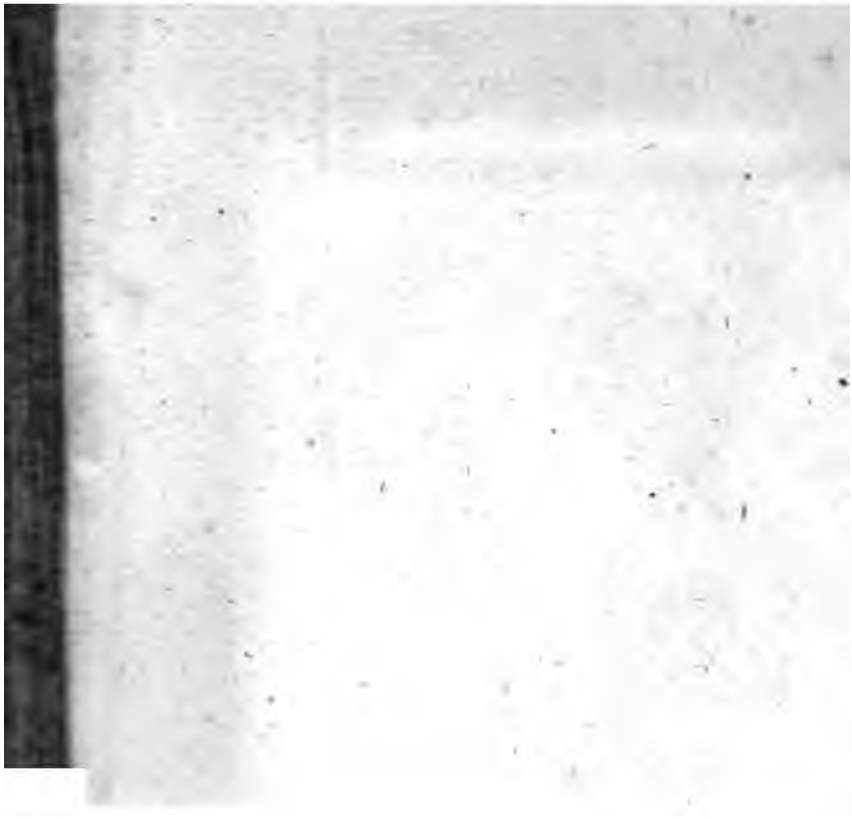

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